MOTIVIC CHERN CLASSES AND IWAHORI INVARIANTS OF PRINCIPAL SERIES

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Abstract. In this expository note, we review the proof [AMSS19] of conjectures of Bump, Nakasuji, and Naruse about principal series representations of p-adic groups. The ingredients of the proof involve Maulik–Okounkov K-theoretic stable basis for the Springer resolution, and motivic Chern classes of Schubert cells for the Langlands dual group

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1. Introduction

The Iwahori–Hecke algebra of a p-adic reductive group is the convolution algebra on the compactly supported functions on the rational points of the group, which are left and right invariant under the Iwahori subgroup. It contains the finite Hecke algebra and the group algebra of the character lattice of the complex dual maximal torus, which is called the lattice part of the Iwahori–Hecke algebra. The Iwahori invariant subspace of any unramified principal series representation is a module over the Iwahori–Hecke algebra via convolution. In the invariant subspace, there is a natural basis induced by the cell decomposition of the group. The elements in this basis are just characteristic functions of the cells. If the unramified character is regular, then the invariant subspace is a regular representation of the finite Hecke algebra, and this basis is the standard basis. However, this basis does not behave well under the intertwiners, making it difficult to compute the corresponding Iwahori–Whittaker functions.

In Casselman's study of unramified principal series [Cas80], he introduced another basis, which is called the Casselman basis now, in the Iwahori invariant subspace. The Casselman basis enjoys many good properties. For example, this is an eigenbasis for the lattice part in the Iwahori–Hecke algebra, and it interacts nicely with the intertwiners. Using this new basis, Casselman proved Macdonald's formula for the spherical function [Mac68], and he also established the formula for the spherical Whittaker function with Shalika [CS80], which is called the Casselman–Shalika formula. Moreover, there are simple formulas for the Whittaker function for the Casselman basis elements [Ree93].

Therefore, to compute the Iwahori–Whittaker functions, we only need to know the transition matrix between these two bases. Special entries of this matrix can be deduced from the Gindikin–Karpelevich formula proved by Langlands [Lan71] in the p-adic setting. Motivated by this formula, Bump and Nakasuji [BN11, BN19] proposed two conjectures about certain matrix coefficients assuming the Dynkin diagram of the group is simply laced. One of them is about the factorization property, while the other one is about the analyticity property of the coefficients. On the other hand, Nakasuji and Naruse [NN16] use Yang–Baxter basis to give a combinatorial formula for every entry of this matrix. However, the formula is too complicated, and we do not know how to use it to prove the conjectures of Bump and Nakasuji.

The difficulty in the factorization conjecture lies in the fact that the condition in the conjecture is about the triviality of certain Kazhdan–Lusztig polynomials [KL79], which is quite hard to work with combinatorially. However, if the Dynkin diagram is simply laced, it is well known that the triviality of a Kazhdan–Lusztig polynomial is equivalent to the condition that the Schubert variety in the complex dual flag manifold is smooth at certain torus fixed point [BL00]. Based on this observation, Naruse refined the factorization conjecture, so that the refined conjecture can be thought of as a smoothness criterion for the Schubert varieties using p-adic representations. In Naruse's refinement, the Dynkin diagram of the group is no longer assumed to be simply laced, and it is an if and only if statement. One of the directions was claimed by Naruse [Nar14].

In this refined factorization conjecture, one side is about p-adic representations, while the other side is about Schubert varieties for the complex dual groups. The connection between these two sides is well studied in geometric representation theory. For example, Kazhdan and Lusztig used equivariant K-theory to prove the Deligne–Langlands conjecture [KL87]. Recall the Iwahori–Hecke algebra is defined as certain functions on the p-adic group. Kazhdan and Lusztig gave another geometric realization of the Iwahori–Hecke algebra via the equivariant K-theory of the Steinberg variety, which is a convolution algebra, see also [CG09]. Moreover, these are categorified by R. Bezrukavnikov in [Bez16].

The proof in [AMSS19] can be thought of as a shadow of these two geometric realizations of the Iwahori–Hecke algebra. To be more specific, the strategy is to consider two geometric realizations of the regular representation of the finite Hecke algebra. On one hand, we have the Iwahori invariants of a regular unramified principal series representation. On the other hand, we have the (specialized) equivariant K-theory of the flag variety for the complex dual group, on which the lattice part subalgebra in the Iwahori–Hecka algebra acts by tensoring by line bundles on the flag variety. By the localization theorem in equivariant K-theory, there is a natural basis in the localized equivariant K-theory of the flag variety–the structure sheaf of the fixed points, and it is immediate to see that they form an eigenbasis for the lattice part of the Iwahori–Hecke algebra. Thus, they correspond to the Casselman basis.

The other basis in the equivariant K-theory corresponding to the standard basis in the p-adic side is called the (dual) motivic Chern classes of the Schubert cells. To generalize the Chern–Schwartz–MacPherson classes for singular varieties in homology [Mac74, Sch65a, Sch65b], Brasselet, Schürmann and Yokura [BSY10] introduce motivic Chern classes in K-theory. See [AMSS19, FRW21] for the equivariant case. The motivic Chern classes of the

Schubert cells form another basis in the equivariant K-theory of the flag variety. Under the above two geometric realizations of the regular representation of the finite Hecke algebra, the standard basis corresponds to the (dual) motivic Chern classes.

Via these correspondences, we can use the equivariant K-theory of the flag variety to give a geometric interpretation of the transition matrix coefficients on the p-adic side. I.e., the coefficients are related to the localization of the motivic Chern classes of the Schubert cells. Finally, the factorization conjecture follows since the motivic Chern classes of Schubert cells contain information about the singularity of the Schubert varieties by its very definition. To be more specific, Kumar [Kum96] has a smoothness criterion for the Schubert varieties using the localization of the Schubert classes in the equivariant cohomology of the flag variety. We generalize it using the localization of the motivic Chern classes.

The analyticity conjecture is proved by proving an analog of it on the flag variety side, which follows from a GKM type argument.

This note is structured as follows. In Section 2, we introduce the objects in the p-adic side: the Iwahori invariant subspace, the two bases, and conjectures of Bump, Nakasuji, and Naruse. In Section 3, we introduce the motivic Chern classes in the complex dual side. Lastly, we sketch the proof of the conjectures in Section 4.

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Notation. Let F be a non-Archimedean local field, with ring of integers $\mathcal{O}_F$, uniformizer $\varpi \in \mathcal{O}_F$, and residue field $\mathbb{F}_q$. Examples are finite extensions of the field of p-adic numbers $\mathbb{Q}_p$, or of the field of Laurent series over $\mathbb{F}_p$. Let G be a split reductive group defined over $\mathcal{O}_F$. Let $B = TN$ be a Borel subgroup containing a maximal torus T and its unipotent radical N. Let R^+ denote the roots in B. We will use α, β (resp. $\alpha^\vee, \beta^\vee$) to denote roots (resp. coroots), and use $\alpha > 0$ to denote $\alpha \in R^+$. Let W be the Weyl group, and $\leq$ denote the Burhat order on it. Let w_0 be the longest element in the Weyl group W.

Let $T^\vee \subset B^\vee \subset G^\vee$ be the complex Langlands dual groups, and $B^{\vee,-}$ be the opposite Borel subgroup. Let $\mathcal{B}$ denote the flag variety $G^\vee/B^\vee$. For any $w \in W$, let $X(w)^\circ := B^\vee w B^\vee/B^\vee \subset G^\vee/B^\vee$ and $Y(w)^\circ := B^{\vee,-} w B^\vee/B^\vee \subset G^\vee/B^\vee$ be the (opposite) Schubert cells, with closures denoted by $X(w)$ and $Y(w)$, respectively. Let $X^*(T^\vee) = X_*(T)$ be the group of characters of $T^\vee$ (=the group of cocharacters of T), and $X_*(T^\vee) = X^*(T)$ be the group of cocharacters of $T^\vee$ (= the group of characters of T). For any $\lambda \in X^*(T^\vee)$, let $\mathcal{L}_\lambda := G^\vee \times_{B^\vee} \mathbb{C}_\lambda$ denote the line bundle over the flag variety $\mathcal{B}$.

2. p-adic side

In this section, we introduce two bases in the Iwahori invariants of the principal series representations of a split reductive p-adic group G, and state the conjectures of Bump, Nakasuji, and Naruse about the transition matrix coefficients between these two bases.

2.1. **Iwahori invariants of the principal series representations.** In this section, we introduce the unramified principal series representation.

Let $G(F)$ be the F-points of G, and similarly for the maximal torus $T(F)$ and Borel subgroup $B(F) = T(F)N(F)$. Let I be an Iwahori subgroup, i.e., the inverse image of $B(\mathbb{F}_q)$ under the natural map $G(\mathcal{O}_F) \to G(\mathbb{F}_q)$.

Let $\mathbb{H} = \mathbb{C}_c[I\backslash G(F)/I]$ be the Iwahori–Hecke algebra, consisting of compactly supported functions on $G(F)$ which are bi-invariant under I. As a vector space, $\mathbb{H} = \Theta \otimes_{\mathbb{C}} H_W(q)$, where Θ is a commutative subalgebra isomorphic to the coordinate ring $\mathbb{C}[T^\vee]$ of the complex dual torus $T^\vee := \mathbb{C}^* \otimes X^*(T)$, and where $H_W(q)$ is the finite Hecke (sub)algebra with parameter q associated to the finite Weyl group W. The finite Hecke algebra $H_W(q)$ is also a subalgebra of $\mathbb{H}$, and it is generated by elements T_w $(w \in W)$ such that the following relations hold: $T_uT_v = T_{uv}$ if $\ell(uv) = \ell(u) + \ell(v)$, and $(T_{s_i} + 1)(T_{s_i} - q) = 0$ for a simple reflection s_i in W.

For any character τ of T, and $\alpha^\vee$ a coroot define $e^{\alpha^\vee}$ by $e^{\alpha^\vee}(\tau) = \tau(h_{\alpha^\vee}(\varpi))$, where $h_{\alpha^\vee} : F^\times \to T(F)$ is the one parameter subgroup. There is a pairing

$$\langle \cdot, \cdot \rangle : T(F)/T(\mathcal{O}_F) \times T^\vee \to \mathbb{C}^*$$

given by $\langle a, z \otimes \lambda \rangle = z^{\mathrm{val}(\lambda(a))}$. This induces an isomorphism between $T(F)/T(\mathcal{O}_F)$ and the group $X^*(T^\vee)$ of rational characters of $T^\vee$. It also induces an identification between $T^\vee$ and unramified characters of $T(F)$, i.e., characters which are trivial on $T(\mathcal{O})$.

From now on we take τ to be an unramified character of $T(F)$ such that $e^\alpha(\tau) \neq 1$ for all coroots α, and for which the stabilizer $W_\tau = 1$. The *principal series representation* is the induced representation $I(\tau) := \mathrm{Ind}_{B(F)}^{G(F)}(\tau)$. As a $\mathbb{C}$-vector space, $I(\tau)$ consists of locally constant functions f on $G(F)$ such that $f(bg) = \tau(b)\delta^{\frac{1}{2}}(b)f(g)$ for any $b \in B(F)$, where $\delta(b) := \prod_{\alpha>0} |\alpha^\vee(b)|_F$ is the modulus function on the Borel subgroup. The group $G(F)$ acts by the formula $(\pi(g)f)(h) = f(hg)$, where $g, h \in G(F)$ and $f \in I(\tau)$. The Iwahori–Hecke algebra $\mathbb{H}$ acts through convolution from the right on the Iwahori invariant subspace $I(\tau)^I$ as follows:

$$(f \star h)(g) := \int_{G(F)} f(gx^{-1})h(x)dx,$$

where $f \in I(\tau)^I$ and $h \in \mathbb{H}$. The restriction of this action to $H_W(q)$ is a regular representation. One can pass back and forth between left and right $\mathbb{H}$-modules by using the standard anti-involution ι on $\mathbb{H}$ given by $\iota(h)(x) = h(x^{-1})$, where $h \in \mathbb{H}$ and $x \in G(F)$. For any $w \in W$, $\iota(T_w) = T_{w^{-1}}$ and $\iota(q) = q$, see [HKP10, Section 3.2]. We use π to denote this left action of $\mathbb{H}$ on $I(\tau)^I$.

2.2. Two bases in the Iwahori invariants of the principal series representations.

In this section, we recall the definitions and properties of the two bases in $I(\tau)^I$.

From the decomposition $G(F) = \bigsqcup_{w \in W} B(F)wI$, one obtains the basis of the *characteristic functions* on the orbits, denoted by $\{\varphi_w \mid w \in W\}$[1]. For $w \in W$, the element φ_w is characterized by the two conditions: [Ree92, pg. 319]:

(1) φ_w is supported on $B(F)wI$;
(2) $\varphi_w(bwg) = \tau(b)\delta^{\frac{1}{2}}(b)$ for any $b \in B(F)$ and $g \in I$.

The left action of $\mathbb{H}$ on $I(\tau)^I$ was calculated by Casselman in [Cas80, Thm. 3.4] as follows. For any simple coroot $\alpha_i^\vee$:

$$\pi(T_{s_i})(\varphi_w) = \begin{cases} q\varphi_{ws_i} + (q-1)\varphi_w & \text{if } ws_i < w; \\ \varphi_{ws_i} & \text{if } ws_i > w. \end{cases} \tag{1}$$

The second basis, called *Casselman's basis*, and denoted by $\{f_w \mid w \in W\}$, was defined by Casselman [Cas80, §3] by duality using certain intertwiner operators. For any character τ and $x \in W$, define $x\tau \in X^*(T)$ by the formula $x\tau(a) := \tau(x^{-1}ax)$ for any $a \in T$. Since τ is unramified and it has trivial stabilizer under the Weyl group action, the space

[1] Our φ_w is equal to $\phi_{w^{-1}}$ in [BN11, BN19]

$\mathrm{Hom}_{G(F)}(I(\tau), I(x^{-1}\tau))$ is known to be one dimensional, spanned by an operator $\mathcal{A}_x = \mathcal{A}_x^\tau$ [2] defined by

$$\mathcal{A}_x(\varphi)(g) := \int_{N_x} \varphi(\dot{x}ng)dn,$$

where $\dot{x}$ is a representative of $x \in W$, $N_x = N(F) \cap \dot{x}^{-1}N^-(F)\dot{x}$ where N^- is the unipotent radical of the opposite Borel subgroup B^-; the measure on N_x is normalized by the condition that $\mathrm{Vol}(N_x \cap G(\mathcal{O}_F)) = 1$ [Ree92]. If $x, y \in W$ satisfy $\ell(x)+\ell(y) = \ell(xy)$, then $\mathcal{A}_y^{x^{-1}\tau}\mathcal{A}_x^\tau = \mathcal{A}_{xy}^\tau$. Then there exist unique functions $f_w \in I(\tau)^I$ such that

$$\mathcal{A}_x^\tau(f_w)(1) = \delta_{x,w}. \tag{2}$$

(Again under our conventions, the f_w is denoted by $f_{w^{-1}}$ in [BN11].) For the longest element w_0 in the Weyl group, Casselman showed in [Cas80, Prop. 3.7] that

$$\varphi_{w_0} = f_{w_0}.$$

Reeder [Ree92] calculated the action of $\mathbb{H}$ on the functions f_w: he showed in [Ree92, Lemma 4.1] that the functions f_w are Θ-eigenvectors, and he calculated in [Ree92, Prop. 4.9] the action of $H_W(q)$. To describe the latter, for any coroot $\alpha^\vee$, let

$$c_{\alpha^\vee} = \frac{1 - q^{-1}e^{\alpha^\vee}(\tau)}{1 - e^{\alpha^\vee}(\tau)}.$$

For any simple coroot $\alpha_i^\vee$ and $w \in W$, write

$$J_{i,w} = \begin{cases} c_{w(\alpha_i^\vee)}c_{-w(\alpha_i^\vee)} & \text{if } ws_i > w; \\ 1 & \text{if } ws_i < w. \end{cases}$$

Then, we have

$$\pi(T_{s_i})(f_w) = q(1 - c_{w(\alpha_i^\vee)})f_w + qJ_{i,w}f_{ws_i}. \tag{3}$$

2.3. **Conjectures of Bump, Nakasuji, and Naruse.** It is an interesting problem to study the transition matrix coefficients between the bases $\{\varphi_w\}$ and $\{f_w\}$. On one hand, this will generalize the Gindikin–Karpelevich formula, see Equation (4) below. On the other hand, the Whittaker function associated with the Casselman basis was computed by Reeder [Ree93]. Thus, knowing the transition matrix will enable us to compute the Iwahori-Whittaker functions, see [BBBG19, MSA19] for some other interpretations of these functions.

In this section, we state conjectures of Bump, Nakasuji, and Naruse, regarding factorization and analyticity properties of the transition matrix coefficients between the bases $\{\varphi_w\}$ and $\{f_w\}$.

We follow mainly [BN11], and we recall that we use opposite notations from those in *loc.cit*: our φ_w and f_w are $\phi_{w^{-1}}$ and $f_{w^{-1}}$ respectively in [BN11]. Let

$$\phi_u := \sum_{u \le w} \varphi_w \in I(\tau)^I,$$

and consider the expansion in terms of the Casselman's basis:

$$\phi_u = \sum_w m_{u,w} f_w.$$

Then by the definition of f_w, $m_{u,w} = \mathcal{A}_w(\phi_u)(1)$. It is also easy to see that $m_{u,w} = 0$ unless $u \le w$, see [BN11, Theorem 3.5].

[2]The intertwiner $\mathcal{A}_x$ is related to M_x from [Cas80, BN11] by the formula $\mathcal{A}_x = M_{x^{-1}}$.

When $u = id$, ϕ_{id} is the spherical vector in $I(\tau)$, i.e. the vector fixed by the maximal compact subgroup $G(\mathcal{O}_F)$, and

$$\mathcal{A}_w(\phi_{id})(1) = m_{id,w} = \prod_{\alpha>0, s_\alpha w<w} \frac{1-q^{-1}e^{\alpha^\vee}(\tau)}{1-e^{\alpha^\vee}(\tau)}. \tag{4}$$

This is the *Gindikin–Karpelevich formula*, which in the non-Archimedean setting was proved by Langlands [Lan71] after Gindikin and Karpelevich proved a similar statement for real groups. Casselman gave another proof using his basis f_w, and this plays a crucial role in his computation of Macdonald formula and the spherical Whittaker functions, see [Cas80, CS80]. See also [SZZ20] for an approach using the stable basis and the equivariant K-theory of the cotangent bundle $T^*(G^\vee/B^\vee)$.

2.3.1. *The factorization conjecture.* For any $u \leq w \in W$, let

$$S(u,w) := \{\beta \in R^+ | u \leq s_\beta w < w\}.$$

Based on Equation (4), Bump and Nakasuji made the following conjecture [BN11, BN19].

Theorem 2.1 (Bump–Nakasuji conjecture)**.** *Assume the Dynkin diagram of G is simply laced. For any $u \leq w \in W$, then*

$$m_{u,w} = \prod_{\alpha \in S(u,w)} \frac{1-q^{-1}e^{\alpha^\vee}(\tau)}{1-e^{\alpha^\vee}(\tau)},$$

if the Kazhdan–Lusztig polynomial

$$P_{w_0w^{-1},w_0u^{-1}}(q) = 1.$$

For any finite simple root system (not necessarily simply laced), it is easy to see that $P_{w_0w^{-1},w_0u^{-1}}(q) = 1$ if and only if $P_{w_0w,w_0u}(q) = 1$ (see [AMSS19, Corollary 9.6]). If the Dynkin diagram of G is simply laced, then $P_{w_0w,w_0u}(q) = 1$ is equivalent to the condition that the opposite Schubert variety $Y(u) := \overline{B^{\vee,-}uB^\vee/B^\vee}$ in the dual complex flag manifold $G^\vee/B^\vee$ is smooth at the torus fixed point $wB^\vee \in G^\vee/B^\vee$ [BL00]. Based on this observation, Naruse [Nar14] refined the Bump–Nakasuji conjecture as follows.

Theorem 2.2 (Bump–Nakasuji–Naruse factorization Conjecture)**.** *For any $u \leq w \in W$,*

$$m_{u,w} = \prod_{\alpha \in S(u,w)} \frac{1-q^{-1}e^{\alpha^\vee}(\tau)}{1-e^{\alpha^\vee}(\tau)},$$

if and only if the opposite Schubert variety $Y(u)$ in the dual complex flag manifold $G^\vee/B^\vee$ is smooth at the torus fixed point $wB^\vee$.

Thus, in this refined conjecture, the Dynkin diagram of G is not necessarily simply laced, and it is an if and only if statement. The original conjecture follows from this refined one. The "if" direction was claimed by Naruse in [Nar14].

2.3.2. *The analyticity conjecture.* Consider the expansion

$$\varphi_u := \sum_w r_{u,w} f_w \in I(\tau)^I.$$

Then by definition,

$$r_{u,w} := \sum_{u\leq x\leq w} (-1)^{\ell(x)-\ell(u)} m_{x,w}. \tag{5}$$

These two sets of matrix coefficients $r_{u,w}$ and $m_{u,w}$ depend on the unramified character τ, which can be thought of as a point in the complex dual torus $T^\vee$. Thus, $r_{u,w}$ and $m_{u,w}$ can be regarded as functions on $T^\vee$, and it is easy to see that they are rational functions on $T^\vee$. Theorem 2.2 implies that

$$\prod_{\alpha\in S(u,w)}(1-e^{\alpha^\vee})m_{u,w}$$

are analytic functions on $T^\vee$ (i.e., having no poles) if the condition in the Theorem is satisfied.

In general, Bump and Nakasuji made the following conjecture, see [BN19, Conjecture 1].

Theorem 2.3 (Analyticity conjecture)**.** *Let $u\leq w$ be two Weyl group elements. Then the functions*

$$\prod_{\alpha\in S(u,w)}(1-e^{\alpha^\vee})r_{u,w} \quad , \quad \prod_{\alpha\in S(u,w)}(1-e^{\alpha^\vee})m_{u,w}$$

are analytic on the complex dual torus $T^\vee$.

The goal of this short note is to sketch the proofs of Theorem 2.2 and Theorem 2.3.

3. Complex dual side

The idea of the proof of the above conjectures is to give a geometric meaning, in the Langlands dual side, of the matrix coefficients $m_{u,w}$. In this section, we work over the complex numbers $\mathbb{C}$, and introduce the motivic Chern classes. We give a smoothness criterion for the Schubert varieties using the motivic Chern classes.

3.1. **Definition of Motivic Chern classes.** In this section, we give the definition of the motivic Chern classes, which is a K-theoretic generalization of the Chern–Schwartz–MacPherson classes in homology [Mac74, Sch65a, Sch65b]. The main references for this subsection are [AMSS19, FRW21].

Let X be a quasi-projective, non-singular, complex algebraic variety, with an action of the torus $T^\vee$. Let $K_{T^\vee}(X):=K^0(\mathrm{Coh}_{T^\vee}(X))$ be the $T^\vee$-equivariant K-theory of X, where $\mathrm{Coh}_{T^\vee}(X)$ is the abelian category of $T^\vee$-equivariant coherent sheaves on X (see [CG09] for a good reference for equivariant K-theory). By definition, the equivariant K-group of a point is

$$K_{T^\vee}(\mathrm{pt})=K^0(\mathrm{Coh}_{T^\vee}(\mathrm{pt}))=\mathrm{Rep}(T^\vee)\simeq\mathbb{Z}[T^\vee],$$

where $\mathrm{Rep}(T^\vee)$ is the finite dimensional representation ring of $T^\vee$ and $\mathbb{Z}[T^\vee]$ is the regular functions on $T^\vee$. Since we can tensor any element in $\mathrm{Coh}_{T^\vee}(X)$ by a finite dimensional $T^\vee$ representation, $K_{T^\vee}(X)$ is a module over $K_{T^\vee}(\mathrm{pt})=\mathbb{Z}[T^\vee]$. Let $K_{T^\vee}(X)_{loc}:=K_{T^\vee}(X)\otimes_{K_{T^\vee}(\mathrm{pt})}\mathrm{Frac}\,K_{T^\vee}(\mathrm{pt})$ denote the localized equivariant K-theory of X, where $\mathrm{Frac}\,K_{T^\vee}(\mathrm{pt})$ denotes the fraction field of $K_{T^\vee}(\mathrm{pt})$.

Recall the (relative) motivic Grothendieck group $G_0^{T^\vee}(\mathrm{Var}/X)$ of varieties over X is the free abelian group generated by isomorphism classes $[f:Z\to X]$ where Z is a quasi-projective $T^\vee$-variety and $f:Z\to X$ is a $T^\vee$-equivariant morphism modulo the usual additivity relations

$$[f:Z\to X]=[f:U\to X]+[f:Z\setminus U\to X]$$

for $U\subset Z$ an open invariant subvariety.

Let y be a formal variable. The following theorem is proved by Brasselet, Schürmann and Yokura [BSY10, Thm. 2.1] in the non-equivariant case. The equivariant case is proved in [AMSS19, FRW21].

Theorem 3.1. *There exists a unique natural transformation*

$$MC_y : G_0^{T^\vee}(\mathrm{Var}/X) \to K_{T^\vee}(X)[y]$$

satisfying the following properties:

(1) *It is functorial with respect to $T^\vee$-equivariant proper morphisms of non-singular, quasi-projective varieties.*
(2) *It satisfies the following normalization condition*

$$MC_y[id_X : X \to X] = \lambda_y(T_X^*) := \sum y^i[\wedge^i T_X^*] \in K_{T^\vee}(X)[y].$$

3.2. **Two bases in the localized equivariant K-theory of the flag variety.** In this section, we introduce two bases in the localized equivariant K-group of the flag variety $\mathcal{B} = G^\vee/B^\vee$.

The flag variety $\mathcal{B}$ has a natural action of the maximal torus $T^\vee$, and the fixed points are in one-to-one correspondence with the Weyl group. For any $w \in W$, $wB^\vee$ is the corresponding fixed point, and let $\iota_w : wB^\vee \hookrightarrow \mathcal{B}$ denote the inclusion of the fixed point. For any $\mathcal{F} \in K_{T^\vee}(\mathcal{B})$, let $\mathcal{F}|_w := \iota_w^*(\mathcal{F}) \in K_{T^\vee}(wB^\vee) = K_{T^\vee}(\mathrm{pt}) = \mathbb{Z}[T^\vee]$ denote the restriction of $\mathcal{F}$ to the fixed point $wB^\vee$. Let $T_w\mathcal{B}$ (resp. $T_w^*\mathcal{B}$) denote the tangent (resp. cotangent) space to $\mathcal{B}$ at the fixed point wB. By the localization theorem [CG09, Chapter 5], the localized equivariant K-group $K_{T^\vee}(\mathcal{B})_{loc}$ has a basis $\{[\mathcal{O}_{wB^\vee}] \mid w \in W\}$, where $\mathcal{O}_{wB^\vee}$ denotes the structure sheaf of the fixed point $wB^\vee$.

Recall that $X(w)^\circ$ (resp. $Y(w)^\circ$) denotes Schubert cells $B^\vee wB^\vee/B^\vee$ (resp. $B^{\vee,-}wB^\vee/B^\vee$). Hence we have $[X(w)^\circ \hookrightarrow \mathcal{B}], [Y(w)^\circ \hookrightarrow \mathcal{B}] \in G_0^{T^\vee}(\mathrm{Var}/\mathcal{B})$. Applying the motivic Chern class transformation MC_y to them, we get

$$MC_y(X(w)^\circ) := MC_y([X(w)^\circ \hookrightarrow \mathcal{B}]) \in K_{T^\vee}(\mathcal{B})[y],$$

and

$$MC_y(Y(w)^\circ) := MC_y([Y(w)^\circ \hookrightarrow \mathcal{B}]) \in K_{T^\vee}(\mathcal{B})[y].$$

Since $X(id)^\circ = B^\vee$ is a point, $MC_y(X(id)^\circ) = [\mathcal{O}_{B^\vee}]$.

By [AMSS19, Proposition 7.1(b)], the localization

$$MC_y(X(w)^\circ)|_w = \lambda_y(T_w^*X(w))\frac{\lambda_{-1}(T_w^*\mathcal{B})}{\lambda_{-1}(T_w^*X(w))} \in K_{T^\vee}(\mathrm{pt})[y],$$

where $T_w^*X(w)$ denotes the tangent space to $X(w)$ at its smooth point $wB^\vee$, and $\lambda_{-1}(V) := \sum(-1)^i[\wedge^i V] \in K_{T^\vee}(\mathrm{pt})$ for any $T^\vee$ representation V. On the other hand, the $MC_y(X(w)^\circ)$ is supported on $X(w) \subset \mathcal{B}$. Therefore, the transition matrix between $\{MC_y(X(w)^\circ) \mid w \in W\}$ and the fixed point basis is upper triangular and the diagonal terms are non-zero. Thus, $\{MC_y(X(w)^\circ) \mid w \in W\}$ forms another basis of the localized equivariant K-group $K_{T^\vee}(\mathcal{B})_{loc}$.

Here is the example of $\mathbb{P}^1$.

Example 3.2. *Let $G^\vee = \mathrm{SL}(2,\mathbb{C})$ and let α denote the unique simple root. Then $\mathcal{B} = \mathbb{P}^1$, $X(id)^\circ = 0$ and $X(s_\alpha)^\circ = \mathbb{P}^1\backslash\{0\}$. By definition, $MC_y(X(id)^\circ) = [\mathcal{O}_0]$, and $MC_y(X(s_\alpha)^\circ) = MC_y(\mathbb{P}^1) - MC_y(X(id)) = \lambda_y(T_{\mathbb{P}^1}^*) - [\mathcal{O}_0]$.*

Remark 3.3. *There is another natural basis, the Schubert basis, in the non-localized equivariant K-group $K_{T^\vee}(\mathcal{B})$, i.e., the structure sheaves of the Schubert varieties $X(w)$'s. The transition matrix coefficients between the motivic Chern classes and the Schubert classes are conjectured to enjoy some (sign) positivity properties, see* [AMSS19]*. The cohomological analogue was conjectured by Aluffi and Mihalcea in* [AM09, AM16]*. The Grassmannian case* *is proved by J. Huh* [Huh16]*. The general non-equivariant case is proved in* [AMSS17]*, and the proof also involves the cohomological stable basis of the cotangent bundle* [Su17, MO19]*.*

3.3. **Hecke algebra action.** In this section, we recall the construction of the Hecke algebra action on $K_{T^\vee}(\mathcal{B})$ due to Lusztig [Lus85], and describe this action on the above two bases.

Recall the finite Hecke algebra $H_W(q)$ is generated over $\mathbb{Z}[q,q^{-1}]$ by elements T_w $(w\in W)$ such that $T_uT_v=T_{uv}$ if $\ell(uv)=\ell(u)+\ell(v)$, and $(T_{s_i}+1)(T_{s_i}-q)=0$ for a simple reflection s_i in W. For any simple root $\alpha_i^\vee$ of $G^\vee$, recall $\mathcal{L}_{\alpha_i^\vee}:=G^\vee\times_{B^\vee}\mathbb{C}_{\alpha_i^\vee}$. Let $P_i^\vee$ be the corresponding minimal parabolic subgroup containing $B^\vee$, and let π_i denote the projection $G^\vee/B^\vee\to G^\vee/P_i^\vee$. The BGG operator ∂_i is $\pi_i^*\circ\pi_{i,*}$. Form the following two sets of operators in $\operatorname{End}_{K_{T^\vee}(\mathrm{pt})[y]}K_{T^\vee}(\mathcal{B})[y]$

$$\mathcal{T}_i:=(1+y\mathcal{L}_{\alpha_i^\vee})\partial_i-id,\quad\textit{and}\quad\mathcal{T}_i^\vee:=\partial_i(1+y\mathcal{L}_{\alpha_i^\vee})-id.$$

On $K_{T^\vee}(\mathcal{B})$, we have a non-degenerate pairing $\langle-,-\rangle$ defined by

$$\langle\mathcal{F},\mathcal{G}\rangle:=\chi_{T^\vee}(\mathcal{B},\mathcal{F}\otimes\mathcal{G})\in K_{T^\vee}(\mathrm{pt}),$$

where $\chi_{T^\vee}(-)$ denotes the $T^\vee$ character on the virtual $T^\vee$-representation $\sum_i(-1)^iH^i(\mathcal{B},-)$.

Then we have the following theorem [AMSS19, Lus85].

Theorem 3.4. (1) *The operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are adjoint to each other, i.e., for any $\mathcal{F},\mathcal{G}\in K_{T^\vee}(\mathcal{B})$, $\langle\mathcal{T}_i(\mathcal{F}),\mathcal{G}\rangle=\langle\mathcal{F},\mathcal{T}_i^\vee(\mathcal{G})\rangle$.*

(2) *Sending $T_i\in H_W(q)$ to $\mathcal{T}_i$ (or $\mathcal{T}_i^\vee$), and q to $-y$ defines an action of $H_W(q)$ on $K_{T^\vee}(\mathcal{B})[y]$.*

The second part of the theorem says that the operators $\mathcal{T}_i$ (resp. $\mathcal{T}_i^\vee$) satisfy the relations in the Hecke algebra. Therefore, we can form $\mathcal{T}_w$ and $\mathcal{T}_w^\vee$ for any $w\in W$. The motivic Chern classes of the Schubert cells are described by these operators in the following nice formula.

Theorem 3.5. [AMSS19] *For any $w\in W$ and simple root $\alpha_i^\vee$ such that $ws_i>w$, we have*

$$\mathcal{T}_i(MC_y(X(w)^\circ))=MC_y(X(ws_i)^\circ).$$

In particular,

$$MC_y(X(w)^\circ)=\mathcal{T}_{w^{-1}}([\mathcal{O}_{B^\vee}]).$$

The cohomological statement is proved in [AM16]. Motivated by this and Theorem 3.4(1), we make the following definition.

Definition 3.6. Let $w\in W$. The *dual motivic Chern class* is defined by

$$MC_y^\vee(Y(w)^\circ):=(\mathcal{T}_{w_0w}^\vee)^{-1}(MC_y(Y(w_0)))=(\mathcal{T}_{w_0w}^\vee)^{-1}(\mathcal{O}_{w_0B^\vee})\in K_{T^\vee}(\mathcal{B})[y,y^{-1}].$$

The name of this class is explained by the following theorem, which is the K-theoretic analogue of [AMSS17, Theorem 5.7].

Theorem 3.7. [AMSS19] *For any $u,v\in W$,*

$$\langle MC_y(X(u)^\circ),MC_y^\vee(Y(v)^\circ)\rangle=\delta_{u,v}(-y)^{\ell(u)-\dim\mathcal{B}}\prod_{\alpha>0}(1+ye^{-\alpha^\vee}).$$

Let $\mathcal{D}$ be the Serre duality functor on $K_{T^\vee}(\mathcal{B})$. I.e., for any $\mathcal{F}\in K_{T^\vee}(\mathcal{B})$, let $\mathcal{D}(\mathcal{F}):=\operatorname{RHom}_{\mathcal{O}_\mathcal{B}}(\mathcal{F},\omega_\mathcal{B}^\bullet)$, where $\omega_\mathcal{B}^\bullet=(-1)^{\dim\mathcal{B}}\mathcal{L}_{2\rho^\vee}$ is the dualizing complex of $\mathcal{B}$, and $2\rho^\vee$ is the sum of all the positive roots in $G^\vee$. Extend $\mathcal{D}$ to $K_{T^\vee}(\mathcal{B})[y,y^{-1}]$ by sending y^i to y^{-i}.

Then the class $MC_y^\vee(Y(w)^\circ)$ is related to the original motivic Chern class $MC_y(Y(w)^\circ)$ via the following lemma.

Lemma 3.8. *For any $w \in W$, we have*

$$MC_y^\vee(Y(w)^\circ) = \prod_{\alpha>0}(1+ye^{-\alpha^\vee})\frac{\mathcal{D}(MC_y(Y(w)^\circ))}{\lambda_y(T^*(\mathcal{B}))} \in K_{T^\vee}(\mathcal{B})_{loc}[y,y^{-1}].$$

This lemma is quite technical. Its proof in [AMSS19] uses the Maulik–Okounkov stable basis [MO19, Oko17] in the equivariant K-theory of the cotangent bundle of the flag variety. Let $i:\mathcal{B}\hookrightarrow T^*(\mathcal{B})$ denote the natural inclusion. Then it is shown in [AMSS19, FRW21] that the pullback of the stable basis elements coincide with the motivic Chern classes of the Schubert cells. On the other hand, the definition of the stable basis depends on a choice of a Weyl chamber, and stable bases for the opposite chambers are dual bases to each other. Combining these with Theorem 3.7, we can prove the above lemma.

Finally, the Hecke algebra action on the fixed point basis is given by the following lemma.

Lemma 3.9. [AMSS19] *The action of the operator $\mathcal{T}_i^\vee$ on the fixed point basis $\{[\mathcal{O}_{wB^\vee}]|w\in W\}$ is given by*

$$\mathcal{T}_i^\vee([\mathcal{O}_{wB^\vee}]) = -\frac{1+y}{1-e^{-w\alpha_i^\vee}}[\mathcal{O}_{wB^\vee}] + \frac{1+ye^{w\alpha_i^\vee}}{1-e^{-w\alpha_i^\vee}}[\mathcal{O}_{ws_iB^\vee}].$$

3.4. **Smoothness criterion.** For the smoothness of Schubert varieties, Kumar proved the following theorem.

Theorem 3.10 ([Kum96])**.** *For any $u\leq w\in W$. The opposite Schubert variety $Y(u)\subset\mathcal{B}$ is smooth at $wB^\vee$ if and only if the localization at $wB^\vee$ of the equivariant fundamental class $[Y(u)]\in H^*_{T^\vee}(\mathcal{B})$ is given by:*

$$[Y(u)]|_w = \left(\prod_{\beta>0,u\nleq s_\beta w}\beta^\vee\right)\in H^*_{T^\vee}(\mathrm{pt})=\mathbb{Z}[\mathrm{Lie}\,T^\vee].$$

We can generalize it using the motivic Chern classes in equivariant K-theory.

Theorem 3.11. [AMSS19] *For any $u\leq w\in W$. The opposite Schubert variety $Y(u)$ is smooth at the torus fixed point $wB^\vee$ if and only if*

$$MC_y(Y(u))|_w = \prod_{\alpha>0,ws_\alpha\geq u}(1+ye^{w\alpha^\vee})\prod_{\alpha>0,u\nleq ws_\alpha}(1-e^{w\alpha^\vee}).$$

The if direction follows from the properties of the motivic Chern classes. For the other direction, we can specialize $y=0$, take the Chern character map, and use Kumar's theorem.

4. The proof

In this section, we outline the proof of Theorem 2.2 and Theorem 2.3. The bridge connecting the p-adic side and the complex side is a shadow of the following well known two geometric realizations of the affine Hecke algebra [CG09, Introduction]

$$K_{G^\vee\times\mathbb{C}^*}(\mathrm{St})\simeq\mathbb{H}\simeq\mathbb{C}_c[I\backslash G(F)/I],$$

where St is the Steinberg variety.

As before, τ is an unramified regular character of T, which determines a point in the dual torus $T^\vee$. Let $\mathbb{C}_\tau$ denote the evaluation representation at $\tau\in T^\vee$ of $K_{T^\vee}(\mathrm{pt})=\mathbb{Z}[T^\vee]$.

For any $w\in W$, define

$$b_w := (-1)^{\dim\mathcal{B}-\ell(w)}\prod_{\alpha>0,w\alpha>0}\frac{y^{-1}+e^{-w\alpha^\vee}}{1-e^{w\alpha^\vee}}[\mathcal{O}_{wB^\vee}]\in K_{T^\vee}(\mathcal{B})[y,y^{-1}].$$

The coefficient in front of $[\mathcal{O}_{wB^\vee}]$ is determined by the following identity

$$b_w|_w = MC_y^\vee(Y(w)^\circ)|_w.$$

In particular, for the longest element w_0, $b_{w_0} = MC_y^\vee(Y(w_0)^\circ)$.

We are now ready state the main result which connects the equivariant K-theory of $\mathcal{B}$ to the Iwahori invariants in $I(\tau)$. Endow $K_{T^\vee}(\mathcal{B})[y,y^{-1}]$ with the $H_W(q)$ action by using the operators $\mathcal{T}_w^\vee$.

Theorem 4.1. [AMSS19] *There is a unique left $H_W(q)$-module homomorphism*

$$\Psi : K_{T^\vee}(G^\vee/B^\vee)[y,y^{-1}] \otimes_{K_{T^\vee}(pt)[y,y^{-1}]} \mathbb{C}_\tau \xrightarrow{\sim} I(\tau)^I,$$

such that:

- $y \mapsto -q$;
- $MC_y^\vee(Y(w)^\circ) \otimes 1 \mapsto \varphi_w$;
- $b_w \otimes 1 \mapsto f_w$.

This type of relation between the equivariant K-theory of the flag variety and representation theory of p-adic groups is also studied by Braverman, Kazhdan, and Lusztig [BK99, Lus98]. The case of the cotangent bundle is also studied in [SZZ20]. In [SZZ20], the authors used this connection to give an equivariant K-theoretic interpretation of the Macdonald's formula for the spherical function [Mac68, Cas80] and the Casselman–Shalika formula for the spherical Whittaker function [CS80].

The proof of this theorem is quite simple. We can define the map Ψ using the first two properties. Then using Equation (1) and Definition 3.6, we check that Ψ is a $H_W(q)$-module isomorphism. Finally, using the fact that $f_{w_0} = \varphi_{w_0}$, $b_{w_0} = MC_y^\vee(Y(w_0)^\circ)$, Equation (3), Lemma 3.9 and the fact that Ψ is a $H_W(q)$-module isomorphism, we can check the last property.

Define the dual operator $(-)^\vee$ on $K_{T^\vee}(\mathrm{pt})[y,y^{-1}] = \mathbb{Z}[X^*(T^\vee)][y,y^{-1}]$ by $(e^\lambda)^\vee = e^{-\lambda}$ for any $\lambda \in X^*(T^\vee)$ and $(y^i)^\vee = y^{-i}$ for any $i \in \mathbb{Z}$.

The conjectures of Bump, Nakasuji, and Naruse are about the transition matrix coefficients $m_{u,w}$ between the two bases on the p-adic side, using Theorem 4.1 and Lemma 3.8, we get the following geometric interpretation of it.

Proposition 4.2. *For any $w \geq u \in W$, we have*

$$m_{u,w} = \left(\frac{MC_y(Y(u))|_w}{MC_y(Y(w)^\circ)|_w}\right)^\vee(\tau),$$

where (τ) means the evaluation at $\tau \in T^\vee$ of the rational function $\frac{MC_y(Y(u))|_w}{MC_y(Y(w)^\circ)|_w} \in \mathrm{Frac}(K_{T^\vee}(\mathrm{pt})[y])$.

Notice that $Y(w)$ is smooth at the torus fixed point $wB^\vee$. Hence, we have an explicit formula for the denominator $MC_y(Y(w)^\circ)|_w$ by Theorem 3.11. Finally, the factorization conjecture (Theorem 2.2) follows from this proposition and Theorem 3.11.

For the analyticity conjecture (Theorem 2.3), we only need to prove it for $m_{u,w}$ because of Equation (5) and the fact that $S(x,w) \subset S(u,w)$ for any $u \leq x \leq w$. Using Proposition 4.2, Theorem 2.3 is translated into the following statement.

Proposition 4.3. *For any $w \leq u \in W$, the polynomial $MC_y(Y(w)^\circ)|_u \in K_{T^\vee}(\mathrm{pt})[y]$ is divisible by*

$$\prod_{\alpha>0, u\alpha>0} (1 + ye^{u\alpha^\vee}) \prod_{\alpha>0, w\nleq us_\alpha<u} (1 - e^{u\alpha^\vee}).$$

This geometric property about motivic Chern classes can be easily proved using a GKM type argument, see [AMSS19, Theorem 7.4].

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