Lecture 1.

ref: · representation theory and complex geometry Chriss - Ginzburg.

· Ginzburg's survey.

Overview

① Springer theory for Weyl groups W.

Springer resolution $T^*(G/B) \xrightarrow{\pi} \mathcal{N} \subseteq \mathfrak{g}$

nilpotent cone.

(a classical example of the symplectic resolution)

$$Z := T^*(G/B) \times_{\mathcal{N}} T^*(G/B).$$

Thm 1) $H^{BM}_{top}(Z) \simeq \mathbb{Q}[W]$

2) All irrep W-reps appears in some homology of fibers of π.

Two ways to show this:

a) via convolution in Borel-Moore homology. (chap 3).

b) Sheaf-theoretic method. (chap 8).

(Ginzburg formalism, many other applications as we'll see in the course).

uses perverse sheaves, decomposition theorem, Fourier transform.

② Springer theory for $\mathcal{U}(\mathfrak{sl}_n)$. (chap 4).

. generalization of the above to the case

T^*(partial flags in type A) $\to \mathcal{N}$

· sheaf-theoretic method by Braverman-Gaitsgory.

· geometric proof of the Schur-Weyl duality by Weiqiang Wang

· Further generalization by Nakajima to Kac-Moody algs.

③ Equivariant K-theory & affine Hecke algs. (chap 5, 7, 8).

· Thm (Kazhdan-Lusztig, Ginzburg)

1) Affine Hecke alg $\mathbb{H}$ for the dual group $\simeq K^{eq}(Z)$.

2) classification of simple modules for $\mathbb{H}$.

(Deligne-Langlands conj.)

(Tamely ramified Langlands conj,

$$\left\{ \begin{array}{l} \text{admissable, irr. } {}^L G(\mathbb{Q}_p)\text{-reps containg non-zero} \\ \text{Iwahori fixed vectors} \end{array} \right\} / \sim \qquad \begin{array}{c} V \\ \rotatebox{90}{$\supseteq$} \\ V^{\text{Iwahori}} \end{array}$$

$$\updownarrow \text{ bijection}$$

$$\text{Simple } \mathbb{H}\text{-mods} / \sim$$

)

- Affine quantum groups.

  (Ginzburg - Vasserot, Nakajima, ⋯).

Other applications of the Ginzburg formalism.

a) Kazhdan - Lusztig conj.

(Brylinski - Kashiwara, Beilinson - Bernstein,

use D-modules on the flag variety).

Braverman - Finkelberg - Nakajima, use Zastava spaces.

b). Coulomb branches BFN.

⋮

$SL(n,\mathbb{C})$ - Case.

$G = SL(n,\mathbb{C})$, $B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix} \right\} \leq G$,

Linear alg $\Rightarrow$ $G = \bigsqcup_{w \in S_n} BwB$. Bruhat decomposition.

The flag variety

$$\mathcal{B} := \{ F = (0 \leq F_1 \leq F_2 \leq \cdots \leq F_n = \mathbb{C}^n) \mid \dim F_i = i \} \simeq G/B$$

Define $\tilde{\mathfrak{g}} = \{ (x, F) \in \mathfrak{sl}_n \times \mathcal{B} \mid x(F_i) \leq F_i, \forall i \}$.

Want to construct the following commutative diagram:

$$\begin{tikzcd}
(x,F) \arrow[d, mapsto] & \in & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
x & \in \mathfrak{g} \arrow[dr, "\phi"'] & & \mathbb{C}^{n-1} = \{ (x_i) \in \mathbb{C}^n \mid \sum x_i = 0 \} \leq \mathbb{C}^n \arrow[dl, "\psi"] \\
& & \mathbb{C}^{n-1}/S_n &
\end{tikzcd}$$

take eigenvalues.

$(x, F) \in \tilde{\mathfrak{g}}$, induces a map $x: F_i/F_{i-1} \to F_i/F_{i-1}$

$x_i =$ eigenvalue of this map

$\nu(x, F) := (x_1, \cdots, x_n) \in \mathbb{C}^n$.

Since $\sum x_i = 0$, $\nu(\tilde{\mathfrak{g}}) \subseteq \{(x_1, \cdots, x_n) \mid \sum x_i = 0\} \simeq \mathbb{C}^{n-1} \subseteq \mathbb{C}^n$.

$x \in \mathfrak{g}$ is called semisimple if it can be diagonalized.

regular if $\dim Z_{\mathfrak{g}}(x) = \mathrm{rk}\,\mathfrak{g} = n-1$.

Hence, $x \in \mathfrak{g}$ is semisimple and regular ($x \in \mathfrak{g}^{rs}$)

if it's eigenvalues are different.

<u>Lemma</u>: For any $x \in \mathfrak{g}^{rs}$, the set $\mathcal{B}_x := \mu^{-1}(x)$ consists of

$n! = \# S_n$ points, and $S_n \curvearrowright \mathcal{B}_x$ freely.

Pf: $\mathbb{C}^n = \bigoplus V_i$, $\dim V_i = 1$

eigenspace decomposition w.r.t. x.

Hence, any element in $\mathcal{B}_x$ is of the form

$$F = (V_{i_1} \subseteq V_{i_1} \oplus V_{i_2} \subseteq \cdots \subseteq \mathbb{C}^n).$$

$\Rightarrow \# \mathcal{B}_x = n!$

$w \in S_n$, $\quad w(F) := (V_{w^{-1}(i_1)} \subseteq V_{w^{-1}(i_1)} \oplus V_{w^{-1}(i_2)} \subseteq \cdots \subseteq \mathbb{C}^n)$. $\square$

$\mathfrak{b}$ = Lie alg of B = upper triangular matrices in $\mathfrak{g}$.

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ \mathfrak{g} & & \mathcal{B} \end{tikzcd}$$

projections.

B acts freely on $G \times \mathfrak{b}$ by $b.(g,x) = (gb^{-1}, bxb^{-1})$.

$G \times_B \mathfrak{b} := G \times \mathfrak{b} / B$.

<u>Lemma</u>: The projection $\pi : \tilde{\mathfrak{g}} \to \mathcal{B}$ makes $\tilde{\mathfrak{g}}$ a G-equivariant vector bundle over $\mathcal{B}$ with fiber $\mathfrak{b}$. Moreover,

$$G \times_B \mathfrak{b} \xrightarrow{\sim} \tilde{\mathfrak{g}} \qquad (g, x) \mapsto (gxg^{-1}, gB/B).$$

Pf: F = Standard flag in $\mathbb{C}^n \in \mathcal{B}$,

then $\pi^{-1}(F)$ = upper triangular matrices $= \mathfrak{b}$. $\square$

Lemma: $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is proper.

pf: μ = restriction of the projection $\mathfrak{g} \times \mathcal{B} \to \mathfrak{g}$,

$\mathcal{B}$ is compact. $\square$.

Example: $\mu^{-1}(0) = \mathcal{B}$

$x \in \mathfrak{g}^{rs}$, $\# \mathcal{B}_x = n!$.

Def: $x \in \mathfrak{g}$ is called nilpotent if all its eigenvalues $= 0$.

$\mathcal{N} :=$ nilpotent matrices in $\mathfrak{g}$.

stable under the dilation $\mathbb{C}^*$-action, $\mathcal{N}$ is called the nilpotent cone.

e.g. $n=2$. $\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| \, a^2+bc=0 \right\}$

a quadratic cone in $\mathbb{C}^3$

$$\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[r, hook] \arrow[d, "\mu"'] & \tilde{\mathfrak{g}} \arrow[d, "\mu"'] \arrow[dr, "\nu"] & \\
\mathcal{N} \arrow[r, hook] \arrow[d] & \mathfrak{g} \arrow[d, "\phi"] & \mathbb{C}^{n-1} = \{(x_i) \in \mathbb{C}^n \mid \sum x_i = 0\} \subseteq \mathbb{C}^n \arrow[dl, "\psi"] \\
\{0\} \arrow[r, hook] & \mathbb{C}^{n-1}/S_n &
\end{tikzcd}$$

Question: when $(x, F) \in \tilde{\mathfrak{g}}$ lies in $\tilde{\mathcal{N}}$.

recall x_i = eigenvalue of x on F_i/F_{i-1}.

if $(x,F) \in \tilde{\mathcal{N}}$, then $x_i = 0$, than $x(F_i) \subseteq F_{i-1}$.

Thus,
$$\tilde{\mathcal{N}} = \{ (x, F) \in \mathfrak{g} \times \mathcal{B} \mid x(F_i) \subseteq F_{i-1} \}.$$

Let $\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix} \right\} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$.

<u>Lemma</u>: 1) $\tilde{\mathcal{N}} \simeq G \times_B \mathfrak{n} \subseteq G \times_B \mathfrak{b} \simeq \tilde{\mathfrak{g}}$

2) $\tilde{\mathcal{N}} \simeq T^*\mathcal{B}$.

pf: 1) follows since $(x, F) \in \tilde{\mathcal{N}} \Rightarrow$ all eigenvalues of $x = 0$.

2). first of all, $T\mathcal{B} \simeq T(G/B) \simeq G \times_B (\mathfrak{g}/\mathfrak{b})$

take duals, $T^*\mathcal{B} \simeq G \times_B (\mathfrak{g}/\mathfrak{b})^* \simeq G \times_B \mathfrak{b}^\perp$

$\mathfrak{b}^\perp$ = orthogonal of $\mathfrak{b} \subseteq \mathfrak{g}^* \underset{\substack{\uparrow \\ \text{trace pairing.}}}{\simeq} \mathfrak{g}$

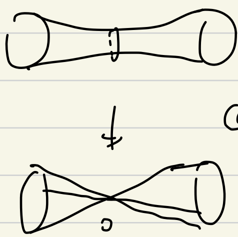
easy exercise shows $\mathfrak{b}^\perp = \mathfrak{n}$.

$\Rightarrow T^*\mathcal{B} \simeq G \times_B \mathfrak{n}$ $\square$.

Thus, $\tilde{\mathcal{N}} := \{(x,F) \mid xF_i \subseteq F_{i-1}\} \simeq G\times_B \mathfrak{n} \simeq T^*\mathcal{B}$

$$\begin{array}{c} \tilde{\mathcal{N}} \\ \mu \downarrow \\ \mathcal{N} \end{array}$$

Ex: $n=2$, $\tilde{\mathcal{N}} = T^*\mathbb{P}^1 \to \mathcal{N}$



$T^*\mathbb{P}^1$

contraction of the zero section $\mathbb{P}^1$

a resolution of singularities

Prop: 1) $\mathcal{N}$ is an irreducible variety of dim $2\dim\mathfrak{n}$.

2) $\exists$ finitely many G-orbits on $\mathcal{N}$

3) $\tilde{\mathcal{N}} \xrightarrow{\mu} \mathcal{N}$ is a resolution of singularity.

Pf: 1) $\mu: T^*\mathcal{B} \to \mathcal{N}$ is surjective, $T^*\mathcal{B}$ is irreducible ($\Leftarrow$ connected & smooth),

$\Rightarrow$ $\mathcal{N}$ is irreducible and $\dim \mathcal{N} \leq \dim T^*\mathcal{B} = 2\dim \mathfrak{n}$.

On the other hand, $x \in \mathfrak{g}$ is nilpotent iff

$$\det(\lambda I - x) = \lambda^n .$$

Hence, $\mathcal{N}$ is cut out by $n-1 = \operatorname{rk}\mathfrak{g}$ equations in $\mathfrak{g}$.

$\Rightarrow$ $\dim \mathcal{N} \geq \dim \mathfrak{g} - \operatorname{rk}\mathfrak{g} = 2\dim \mathfrak{n}$

$\Rightarrow$ $\dim \mathcal{N} = 2\dim\mathfrak{n}$.

2) follows from Jordan decomposition.

$$\mathcal{N} = \bigsqcup_{\lambda \vdash n} \mathcal{O}_\lambda$$

Jordan block of sizes given by λ.

partitions of n

$$\mathcal{O}_{(n)} = G\text{-orbit of } \underset{\substack{\shortparallel \\ x}}{\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}} = \text{regular nilpotent matrices}$$

$(\dim Z_{\mathfrak{g}}(x) = \operatorname{rk}\mathfrak{g})$

$\dim \mathcal{O}_{(n)} = \dim G/Z_G(x) = 2\dim \mathfrak{n} = \dim \mathcal{N}$.

$\mathcal{O}_{(n)}$ = Zariski-open. dense orbit.

3). we show μ is an isomorphism over $\mathcal{O}_{(n)}$.

Suppose $F = (F_1 \subseteq F_2 \subseteq \cdots \subseteq \mathbb{C}^n) \in \mu^{-1}(x)$.

Since $x(F_i) \subseteq F_{i-1} \Rightarrow F_1 = \ker x = \langle e_1 \rangle$,

$F_2 = \ker x^2 = \langle e_1, e_2 \rangle, \cdots$.

Thus, $\mu^{-1}(x)$ consists of one point.

μ is birational. $\tilde{\mathcal{N}} = T^*\mathcal{B}$ is smooth

$\Rightarrow \mu$ is a resolution of singularities. $\square$.