Intersection Cohomology.

Motivation: X nonsingular, irreducible proj variety, $d = \dim_{\mathbb{C}} X$

Poincaré duality $H^i(X, \mathbb{C}_X) \simeq \left(H^{2d-i}(X, \mathbb{C}_X)\right)^*$.

For singular variety, Goresky and Macpherson constructed

$IC_X^{\cdot}[-d] \in \mathrm{Perv}(X)$, and define intersection cohomology groups

$$IH^i(X) := H^i(X, IC_X^{\cdot}[-d]) \quad 0 \le i \le 2d$$

then $\exists$ a generalized Poincaré duality

$$IH^i(X) \simeq \left(IH^{2d-i}(X)\right)^* \quad \text{for any proj. variety } X.$$

From now on, let's assume $X = \sqcup X_\alpha$ is a Whitney stratification.

Minimal extension of perverse sheaves

X irreducible, $U \subseteq X$ Zariski open dense.

$$Z := X \setminus U \xhookrightarrow{i} X \xhookleftarrow{j} U$$

$F^\cdot \in \mathrm{Perv}(U)$, $\exists$ canonical morphism $j_! F^\cdot \to j_* F^\cdot$.

Take ${}^p\mathcal{H}^0$, we get

$$ {}^p j_! F^\cdot \to {}^p j_* F^\cdot \quad \text{in } \mathrm{Perv}(\mathbb{C}_X)$$

<u>Def</u> ${}^p j_{!*} F^\cdot := \mathrm{image}({}^p j_! F^\cdot \to {}^p j_* F^\cdot) \in \mathrm{Perv}(\mathbb{C}_X)$.

<u>Prop</u>: $D_X({}^p j_{!*} F^\cdot) \simeq {}^p j_{!*}(D_U F^\cdot)$

pf: Applying D_X to

$$ {}^p j_! F^\cdot \twoheadrightarrow {}^p j_{!*} F^\cdot \hookrightarrow {}^p j_* F^\cdot, \quad \text{we get}$$

$$D_X({}^p j_* F^\cdot) \twoheadrightarrow D_X({}^p j_{!*} F^\cdot) \hookrightarrow D_X({}^p j_! F^\cdot)$$

Since D_X is t-exact,

$$D_X({}^p j_* F^\cdot) \simeq {}^p H^0 D_X(j_* F^\cdot) = {}^p j_!(D_U F^\cdot)$$

$$D_X({}^p j_! F^\cdot) = {}^p H^0 D_X(j_! F^\cdot) = {}^p j_*(D_U F^\cdot)$$

Hence $\quad {}^p j_!(D_U F^\cdot) \twoheadrightarrow D_X({}^p j_{!*} F^\cdot) \hookrightarrow {}^p j_*(D_U F^\cdot)$

$$\Rightarrow \quad D_X({}^p j_{!*} F^\cdot) \simeq {}^p j_{!*}(D_U F^\cdot) \qquad \square$$

<u>Prop</u>: The minimal extension $G^\cdot = {}^p j_{!*} F^\cdot$ is characterized as the unique perverse sheaf on X satisfying:

(i) $G^\cdot|_U \simeq F^\cdot$

$i: Z \hookrightarrow X \hookleftarrow U : j$

(ii) $i^* G^\cdot \in {}^p D_c^{\leq -1}(Z)$

(iii) $i^! G^\cdot \in {}^p D_c^{\geq 1}(Z)$

Pf: We first show ${}^p j_{!*} F^\cdot$ satisfy these properties.

$j^* = j^!$ is t-exact, hence

$${}^p j_{!*} F^\cdot|_U = j^*\left(\mathrm{Im}({}^p j_! F^\cdot \to {}^p j_* F^\cdot)\right)$$

$$= \mathrm{Im}\left(j^* \, {}^p j_! F^\cdot \to j^* \, {}^p j_* F^\cdot \right)$$

$$= \mathrm{Im}\left({}^pH^0 (j^* j_! F^\cdot) \to {}^pH^0 (j^* j_* F^\cdot) \right)$$

$$= \mathrm{Im} (F^\cdot \to F^\cdot) = F^\cdot$$

$\Rightarrow$ (i).

The dist'd $\triangle$

$$j_! j^* G^\cdot \to G^\cdot \to i_* i^* G^\cdot \xrightarrow{+1} ,$$

We get

$$\begin{tikzcd}
{}^pH^0(j_! j^* G^\cdot) \arrow[r] \arrow[d, equal, "j^*G^\cdot = F^\cdot"] & {}^pH^0(G^\cdot) \arrow[r] \arrow[d, equal] & {}^pH^0(i_* i^* G^\cdot) \arrow[r] & {}^pH^1(j_! j^* G^\cdot) \arrow[r] \arrow[d, equal] & {} \\
{}^p j_! F^\cdot \arrow[r, two heads] & {}^p j_{!*} F^\cdot & & {}^pH^1(j_! F^\cdot) \arrow[d, equal, "j_! \text{ is right t-exact.}"] & \\
& & & 0 &
\end{tikzcd}$$

$\Rightarrow {}^pH^0 (i_* i^* G^\cdot) = 0$

$$\left.\begin{array}{l} \xRightarrow[\text{t-exact}]{i_* = i_! \text{ is}} {}^pH^0 (i^* G^\cdot) = 0 \\ i^* \text{ is right t-exact} \Rightarrow i^* G^\cdot \in {}^pD_c^{\le 0}(Z) \end{array}\right\} \Rightarrow i^* G^\cdot \in {}^pD_c^{\le -1}(Z)$$

$\Rightarrow$ condition (ii)

Exercise: use $i_* i^! G^\cdot \to G^\cdot \to j_* j^* G^\cdot \xrightarrow{+1}$ to prove (iii).

Finally, let's show that $G^\cdot \in Perv(\mathbb{C}_X)$ satisfying (i), (ii), (iii) is canonically isomorphic to ${}^p\bar{j}_{!*}F^\cdot$.

Since $\bar{j}^! G^\cdot = F^\cdot = \bar{j}^* G^\cdot$, we get

$$\bar{j}_! F^\cdot \to G^\cdot \to \bar{j}_* F^\cdot \text{ by adjunction.}$$

$$\Rightarrow {}^p\bar{j}_! F^\cdot \to G^\cdot \to {}^p\bar{j}_* F^\cdot \text{ in } Perv(\mathbb{C}_X)$$

Hence, enough to show ${}^p\bar{j}_! F^\cdot \to G^\cdot$ is an epimorphism, and

$G^\cdot \to {}^p\bar{j}_* F^\cdot$ is a monomorphism in $Perv(\mathbb{C}_X)$.

Let's show the former.

The cokernel of ${}^p\bar{j}_! F^\cdot \to G^\cdot$ is supported on Z,

$\exists$ an exact sequence

$${}^p\bar{j}_! F^\cdot \to G^\cdot \to i_* H^\cdot \to 0 \quad \text{for some } H \in Perv(\mathbb{C}_Z) \simeq Perv_Z(\mathbb{C}_X)$$

Since i^* is right t-exact, we get

($\Rightarrow$ ${}^p i^*$ is right exact)

$$
\begin{array}{c}
{}^p i^* G^\cdot \to {}^p i^* i_* H^\cdot \to 0 \\
\| \\
H^\cdot
\end{array}
$$

$i^* G \in {}^p D_c^{\le -1}(Z) \Rightarrow {}^p i^* G = 0 \Rightarrow H = 0$

$\Rightarrow {}^p j_! F^\cdot \to G^\cdot$ is an epimorphism. $\square$

<u>Cor</u> X smooth, for any $\mathcal{L} \in \mathrm{Loc}(X)$,

$$\mathcal{L}[d_X] \simeq {}^p j_{!*}(\mathcal{L}|_U[d_X])$$

pf: only need to show $i^* \mathcal{L}[d_X] \in {}^p D_c^{\le -1}(Z)$

$$\Uparrow$$
$$d_Z < d_X.$$

and $i^! \mathcal{L}[d_X] \in {}^p D_c^{\ge 1}(Z)$.

$$
\begin{array}{l}
\| \\
i^! D_X^2 \mathcal{L}[d_X] = D_Z i^*(\mathcal{L}^*[d_X]) \in {}^p D_c^{\le 1}(Z)
\end{array}
$$
$\square$

Prop: $F^\cdot \in Perv(\mathbb{C}_U)$, then

i) ${}^p j_* F^\cdot$ has no non-trivial subobj. supported on Z.

ii) ${}^p j_! F^\cdot$ has no non-trivial quotient obj. supp. on Z.

pf: i) $0 \to G^\cdot \to {}^p j_* F^\cdot$, $\quad G^\cdot \in Perv_Z(\mathbb{C}_X)$

$$\begin{array}{ccc} Perv_Z(\mathbb{C}_X) & \underset{{}^p i^* = {}^p i^!}{\overset{i_* = i_!}{\rightleftarrows}} & Perv(\mathbb{C}_Z) \\ \cup & & \\ G^\cdot & & \end{array}$$

$G^\cdot \simeq i_* {}^p i^! G^\cdot$.

$i^!$ is left t-exact, then (Prop 8.1.15 in [HTT])

a) $\tau^{\le 0} i^! \tau^{\le 0} \simeq \tau^{\le 0} i^!$

b) ${}^p i^!$ is a left exact functor.

By b), apply ${}^p i^!$ to $0 \to G^\cdot \to {}^p j_* F^\cdot$ in $Perv(\mathbb{C}_X)$

we get $\quad 0 \to {}^p i^! G^\cdot \to {}^p i^! {}^p j_* F^\cdot$

since j_* is also left t-exact.

$$ {}^p i^! \, {}^p j_* F^\cdot = \tau^{\leq 0} i^! \tau^{\leq 0} j_* F^\cdot \underset{a)}{\simeq} \tau^{\leq 0} i^! j_* F^\cdot \underset{\uparrow}{=} 0 $$

$$ i^! j_* F^\cdot = 0 $$

$\square$

<u>Cor</u> The minimal extension ${}^p j_{!*} F^\cdot$ has neither non-trivial subobj. nor non-trivial quotient object whose supp. is contained in Z.

<u>Cor</u>: Assume $F^\cdot \in \mathrm{Perv}(\mathbb{C}_U)$ is simple, then ${}^p j_{!*} F^\cdot$ is also a simple object.

Pf: Let $G^\cdot \subseteq {}^p j_{!*} F^\cdot$ be a sub object in $\mathrm{Perv}(\mathbb{C}_X)$

$$ 0 \to G^\cdot \to {}^p j_{!*} F^\cdot \to H^\cdot \to 0. $$

$j^! = j^*$ is t-exact. Hence, it's also an exact functor on $\mathrm{Perv}(\mathbb{C}_X)$,

$$0 \to j^! G^\cdot \to F^\cdot \to j^! H^\cdot \to 0.$$

F simple $\Rightarrow$ $j^! G^\cdot = 0$ or $j^! H^\cdot = 0$.

i.e. either G or H is supp. on Z.

$\Rightarrow$ either G or H is 0. $\square$

<u>Cor</u> (Perverse continuation principle).

Any morphism $\mathcal{L}_1 \to \mathcal{L}_2$ of local systems on U can be uniquely extended to a morphism ${}^p j_{!*}\mathcal{L}_1 \to {}^p j_{!*}\mathcal{L}_2$ of the minimal extensions, this gives an isomorphism

$$\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) \xrightarrow{\sim} \mathrm{Hom}({}^p j_{!*}\mathcal{L}_1, {}^p j_{!*}\mathcal{L}_2)$$

Truncation formula

$X = \bigsqcup X_\alpha$ Whitney stratification.

$$X_k := \bigsqcup_{\dim X_\alpha \le k} X_\alpha ,$$

$$X \supseteq X_{d_X} \supseteq X_{d_X - 1} \supseteq \cdots \supseteq X_0 \supseteq X_{-1} = \emptyset .$$

$U_k := X \setminus X_{k-1}$.

$$\begin{tikzcd}
U := U_{d_X} \arrow[r, hook, "j_{d_X}"] \arrow[rrrr, bend right, "j"'] & U_{d_X - 1} \arrow[r, hook] & \cdots \arrow[r, hook] & U_1 \arrow[r, hook, "j_1"] & U_0 = X
\end{tikzcd}$$

<u>Prop</u>: $\forall\, \mathcal{L} \in \mathrm{Loc}(U)$

$$^p j_{!*}(\mathcal{L}[d_X]) \simeq \left(\tau^{\le -1} j_{1*}\right) \circ \cdots \circ \left(\tau^{\le -d_X} j_{d_X *}\right)(\mathcal{L}[d_X])$$

<u>Lemma</u>: Let $U' \supseteq U$, open.

$$\begin{tikzcd}
U \arrow[r, hook, "j_1"] \arrow[rr, bend right, "j"'] & U' \arrow[r, hook, "j_2"] & X
\end{tikzcd}$$

then (i) $^p j_* \simeq {}^p j_{2*}\, {}^p j_{1*}$, $\;{}^p j_! \simeq {}^p j_{2!} \circ {}^p j_{1!}$ (ii) $^p j_{!*} F \simeq {}^p j_{2!*} \circ {}^p j_{1!*} F$.

Pf of the prop: By the lemma, we only need to show:

for any $F^\cdot \in \mathrm{Perv}(\mathbb{C}_{U_k})$, whose restriction to any $X_\alpha \subseteq U_k$

has locally constant cohomology sheaves, we have

$$ {}^p\bar{j}_{k!*}F^\cdot \simeq \tau^{\leq -k}\bar{j}_{k*}F^\cdot. $$

We show this using the characterizing properties of ${}^p\bar{j}_{k!*}F^\cdot$.

Let $G^\cdot := \tau^{\leq -k}\bar{j}_{k*}F^\cdot$.

Since U_k consists of strata with dim $\geq k$, we get

$$ \mathcal{H}^r(F^\cdot) = 0 \text{ for } r > -k. $$

$\Rightarrow$ $\tau^{\leq -k} R\bar{j}_{k*}(F^\cdot)\,|_{U_k} \simeq F^\cdot$ Condition i) ✓

Let $Z := U_{k-1} \setminus U_k = \bigsqcup_{\dim X_\alpha = k-1} X_\alpha \xhookrightarrow{i} X$

$G^\cdot := \tau^{\leq -k}\bar{j}_{k*}F^\cdot$, $\mathcal{H}^r(G^\cdot) = 0$ for $r > -k$

$\Rightarrow$ $\mathcal{H}^r(i^*G^\cdot) = 0$ for $r > -k$. $\Rightarrow$ $i^*G^\cdot \in {}^pD_c^{\leq -1}(Z)$

$\Rightarrow$ Condition ii) ✓.

Consider the dist'd Δ,

$$G^\cdot \longrightarrow j_{k*}F^\cdot \longrightarrow \tau^{\geq -k+1} j_{k*}F^\cdot \xrightarrow{+1},$$

Apply $i^!$, use $i^! j_{k*}F^\cdot \simeq 0$, we get

$$i^! G^\cdot \simeq i^!(\tau^{\geq -k+1} j_{k*}F^\cdot)[-1]$$

$\Rightarrow \mathcal{H}^r(i^! G^\cdot) = 0$ for $r \leq -k+1$.

$\Rightarrow i^! G^\cdot \in {}^p D_c^{\geq 1}(Z)$ $\Rightarrow$ condition iii) ✓ $\square$

<u>Example</u>: X smooth curve$/\mathbb{C}$. $X = U \sqcup \{x_1\} \sqcup \{x_2\} \sqcup \cdots \sqcup \{x_m\}$ as before.

$j_*\mathcal{L}[1]$ has stalks

	-2	-1	1	0
U	0	$\mathcal{L}$	0	0
x_i	0	V^u	V_u	0

$\tau^{\leq -1}(j_*\mathcal{L}[1])$ has stalks

	-2	-1	1
U	0	$\mathcal{L}$	0
x_i	0	V^u	0

Suppose the monodromy doesn't have 1 as an eigenvalue,

then $V^u = V_u = \{0\}$

$\Rightarrow j_* \mathcal{L}[1] = j_! \mathcal{L}[1] = {}^p j_{!*}(\mathcal{L}[1])$

<u>Def</u>: 1) For an irreducible variety X, define its intersection cohomology complex $IC_X^\cdot \in \mathrm{Perv}(\mathbb{C}_X)$ by

$$IC_X^\cdot := {}^p j_{!*}(\mathbb{C}_{X_{reg}}[d_X]).$$

$j: X_{reg} \hookrightarrow X.$

Then $\mathbb{D}_X(IC_X^\cdot) = IC_X^\cdot.$

2) intersection cohomology groups

$$IH^i(X) := H^i(X, IC_X^\cdot[-d_X])$$

$$IH_c^i(X) := H_c^i(X, IC_X^\cdot[-d_X])$$

Thm (generalized Poincaré duality)

$$IH^i(X) \simeq \left(IH_c^{2d_X-i}(X) \right)^* \qquad 0 \le i \le 2d_X$$

pf: $p: X \to pt$,

$$p_* IC_X[-d_X] = D_{pt}\, p_!\, D_X\left(IC_X[-d_X]\right)$$

$$= \left(p_!\, IC_X[d_X]\right)^*$$

$$\Rightarrow IH^i(X) = H^i(p_* IC_X[-d_X])$$

$$= \left(H^{-i}(p_! IC_X[d_X])\right)^*$$

$$= \left(IH_c^{2d-i}(X)\right)^* \qquad \square$$