Decomposition theorem.

Def: $Y \subseteq X$ a locally closed subvariety, $\mathcal{L} \in \mathrm{Loc}(U)$.

$U \subseteq (\overline{Y})_{reg}$ Zariski open dense.

Define $IC(Y, \mathcal{L}) := i_* {}^p j_{!*}(\mathcal{L}[d_Y]) \in \mathrm{Perv}(X)$.

where $U \overset{j}{\hookrightarrow} \overline{Y} \overset{i}{\hookrightarrow} X$.

Thm $IC(Y, \mathcal{L})$ satisfies:

a) $\mathcal{H}^i IC(Y, \mathcal{L}) = 0 \quad i < -d$.

b) $\mathcal{H}^{-d} IC(Y, \mathcal{L})|_U = \mathcal{L}$

c) $\dim \operatorname{supp} \mathcal{H}^i IC(Y, \mathcal{L}) < -i \quad$ if $i > -d$

d) $\dim \operatorname{supp} \mathcal{H}^i ((IC(Y, \mathcal{L}))^\vee) < -i \quad$ if $i > -d$

e) $\mathcal{H}^{-d} IC(Y, \mathcal{L}) = \mathcal{H}^0(j_* \mathcal{L})$

f) if $\mathcal{L}$ is simple, then $IC(Y, \mathcal{L})$ is a simple object in $\mathrm{Perv}(\mathbb{C}_X)$.

Rmk: a), b), c), d) follows from the characterization of the minimal extensions $Z := X \setminus U \overset{k}{\hookrightarrow} X$

$k^* IC(Y, \mathcal{L}) \in {}^p D_c^{\leq -1}(Z) \Rightarrow c)$

The truncation formula $\Rightarrow$ e.

Thm ([BBDG])

The simple objects of $\mathrm{Perv}(\mathbb{C}_X)$ are the intersection cohomology complex $IC(Y, \mathcal{L})$, $Y \subseteq X$ locally closed irreducible, $\mathcal{L}$ simple local systems

Cor: a) $\mathrm{Ext}^k(IC(Y,\mathcal{L}), IC(Y',\mathcal{L}')) = 0 \quad \forall k < 0$

b) If $\mathcal{L}, \mathcal{L}'$ are irreducible,

$$\mathrm{Ext}^0(IC(Y,\mathcal{L}), IC(Y',\mathcal{L}')) = \begin{cases} \mathbb{C} & \text{if } Y = Y', \mathcal{L} = \mathcal{L}' \\ 0 & \text{otherwise.} \end{cases}$$

Thm (Decomposition theorem, [BBDG])

$\mu: M \to N$ proper, then

$$\mu_* IC(M) = \bigoplus_{(i,Y,\chi)} L_{Y,\chi}(i) \otimes IC(Y,\chi)[i] \in D^b_c(N)$$

where Y runs over locally closed subvarieties of N, χ is an irreducible local system on some dense open smooth subset of Y, $[i]$ = degree shift, and $L_{Y,\chi}(i)$ are certain finite dim'l vector spaces.

Rmk: M smooth $/\mathbb{C}$, $\mu: M \to N$ projective morphism, Then $\exists$ a stratification $N = \coprod_\alpha N_\alpha$, s.t. $\forall \alpha$,

$\mu: \mu^{-1}(N_\alpha) \to N_\alpha$ is a locally trivial fibration.

Thus, the theorem takes the following form

$$\mu_* \mathbb{C}_M[\dim_{\mathbb{C}} M] = \bigoplus_{\substack{k \in \mathbb{Z} \\ \phi = (N_\beta, \chi_\beta)}} L_\phi(k) \otimes IC_\phi[k].$$

Where IC_ϕ is the intersection cohomology complex ass. to an irreducible local system χ_β on N_β. (N_β is smooth.)

Example: Springer resolution for $G = SL(2, \mathbb{C})$

$X := T^* \mathbb{P}^1 \xrightarrow{\mu} N = \{(a,b,c) \mid ab = c^2\}$.

$$\mu_* \mathbb{C}_X[2]\Big|_y = \begin{cases} \mathbb{C}[2] & \text{of } y \in N \\ \mathbb{C}[0] \oplus \mathbb{C}[2] & y = 0 \end{cases}$$

$N^* := N \setminus \{0\}$. $\quad \mu_* \mathbb{C}_X[2]\big|_{N^*} = \mathbb{C}_{N^*}[2]$.

$\Rightarrow IC(N, \mathbb{C}_{N^*}[2])$ is a direct summand of $\mu_* \mathbb{C}_X[2]$

Since $\mathcal{H}^0 IC(N, \mathbb{C}_{N^*}[2]) = 0$.

$\Rightarrow \; IC(N, \mathbb{C}_{N^*}[2])\big|_0 = \mathbb{C}[2]$.

Let $\mu_* \mathbb{C}_X[2] = IC(N, \mathbb{C}_{N^*}[2]) \oplus \mathcal{F}$.

then $\mathcal{F}|_y = \begin{cases} 0 & y \in N^* \\ \mathbb{C}[0] & y = 0 \end{cases}$

$\Rightarrow \mathcal{F} = \mathbb{C}_0$,

Moreover, $IC(N, \mathbb{C}_{N^*}[2]) = \mathbb{C}_N[2]$

and $\mu_* \mathbb{C}_X[2] = \mathbb{C}_N[2] \oplus \mathbb{C}_0$.

(we will revisit this later)

Equivariant Version.

G alg group. $X = G/H$ a homogeneous space. $H \subseteq G$.

We can talk about G-equiv. local systems on X.

$H \hookrightarrow G \to G/H$ gives

$$\pi_1(G) \to \pi_1(X) \to \pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to 1$$

$H^\circ \subseteq H$ the identity component.

$$\pi_1(G/H) \twoheadrightarrow H/H^\circ$$

<u>Lemma</u> $\mathcal{L} \in Loc(G/H)$ is G-equivariant iff the corresponding rep of $\pi_1(G/H, x)$ on $\mathcal{L}_x$ is the pullback of a finite dim'l rep of H/H° from the map $\pi_1(G/H, x) \to H/H^\circ$.

Now assume $G \circlearrowright M, N$, M smooth

$\mu : M \to N$ is G-equivariant, projective

$N = \sqcup \mathcal{O}$ finitely many G-orbits.

Thm:
$$\mu_* \mathbb{C}_M[d_M] = \bigoplus_{\substack{i \in \mathbb{Z} \\ \phi = (\mathcal{O}, \chi)}} L_\phi(i) \otimes IC_\phi[i],$$

where $\phi = (\mathcal{O}, \chi)$ runs over: a G-orbit $\mathcal{O}$ on N,

an irreducible G-equiv. local system χ on $\mathcal{O}$; $[i]$ = shift,

$L_\phi(i)$ are certain finite dim'l vector spaces.

Rmk: for $\phi = (\mathcal{O}, \chi)$, choose $x_\phi \in \mathcal{O} \simeq G/G_{x_\phi}$,

Then the local system χ on $\mathcal{O}$ corresponds to a

finite dim'l rep of $G_{x_\phi}/G_{x_\phi}^\circ$.

(Semi)small maps.

<u>Def</u>: $\mu : M \to N$ dominant morphism of irreducible varieties,

We say μ is small (resp. semismall) if

$$\operatorname{codim}_N \{n \in N \mid \dim \mu^{-1}(n) \geq k\} > 2k$$

$$(\text{resp. } \operatorname{codim}_N \{n \in N \mid \dim \mu^{-1}(n) \geq k\} \geq 2k)$$

for any $k \geq 1$.

<u>Rmk</u>: 1) μ semismall, then $\exists$ open dense subset $U \subseteq N$ st.

$\mu|_{\mu^{-1}(U)} : \mu^{-1}(U) \to U$ is a finite morphism.

In particular, $d_M = d_N$.

2) In terms of the above stratification $N = \bigcup N_\alpha$,

$x \in N_\alpha$, $M_x := \mu^{-1}(x)$,

μ is semismall if $\dim N_\alpha + 2\dim M_x \leq \dim M$.

μ is small if $\dim N_\alpha + 2\dim M_x < \dim M$ for all N_α, st. $\overline{N_\alpha} \neq N$.

Def. 1) We say a stratum N_α is relevant if $2\dim M_x = \operatorname{codim} N_\alpha$.

2) We say a pair (N_α, χ_α) is relevant if N_α is relevant, and the irreducible local system χ_α appears in the local system on N_α whose stalks are $H_{2\dim M_x}(M_x)$.

Rmk: this local system can also be described as the dual of the local system $\mathcal{H}^{-\dim N_\alpha} \mu_* \mathbb{C}_M[d]|_{N_\alpha}$.

$$\begin{tikzcd} M_x \arrow[r] \arrow[d] & M \arrow[d, "\mu"] \\ x \arrow[r, hook, "i"'] & N \end{tikzcd}$$

$\forall x \in N_\alpha$, $\mathcal{H}_x^{-\dim N_\alpha} \mu_* \mathbb{C}_M[d]|_x$

$= \mathcal{H}_x^{2\dim M_x}(i^* \mu_! \mathbb{C}_M)$

$= H_c^{2\dim M_x}(M_x, \mathbb{C}_{M_x})$

$= H^{2\dim M_x}(M_x)$

$\simeq \left(H_{2\dim M_x}(M_x)\right)^*.$

Thm (Borho–MacPherson)

1) if μ is semismall, then $\mu_*\mathbb{C}_M[d_M]$ is a perverse sheaf.

$$\mu_*\mathbb{C}_M[d] = \bigoplus_{\phi=(N_\phi,\chi_\phi)} L_\phi \otimes IC_\phi$$

2) Only the relevant pairs $\phi = (N_\phi, \chi_\phi)$ appears in the decomposition thm. Moreover, for such a pair, and $x \in N_\phi$.

$$L_\phi = H_{top}(M_x)_\phi = \mathrm{Hom}_{\pi_1(N_\phi,x)}(\chi_\phi, H_{top}(M_x))$$

3) If μ is small, and N is irreducible, then

$$\mu_*\mathbb{C}_M[d] = IC(\mu_*\mathbb{C}_M[d]|_{N_0}),$$

where $N_0 \subseteq N$ is the dense stratum.

Pf: 1) $x \in N$, $i_x : \{x\} \hookrightarrow N$.

$$\mathcal{H}^j(\mu_*\mathbb{C}_M[d])_x = H^j i_x^* \mu_! \mathbb{C}_M[d]$$

$$\begin{tikzcd} M_x \arrow[r, hook] \arrow[d] & M \arrow[d, "\mu"] \\ x \arrow[r, hook, "i"] & N \end{tikzcd}$$

$$= H^{j+d}(M_x),$$

Hence, if $x \in N_\alpha$,

$$\mathcal{H}^j(\mu_* \mathbb{C}_M[d])_x \neq 0 \Rightarrow j+d \leq 2\dim M_x \overset{\text{semismall.}}{\Rightarrow} j+d \leq d - \dim N_\alpha$$

$\Rightarrow \dim N_\alpha \leq -j.$

$\Rightarrow \mu_* \mathbb{C}_M[d] \in {}^pD_c^{\leq 0}(N).$

Since μ is proper, $D_N(\mu_* \mathbb{C}_M[d]) = \mu_* \mathbb{C}_M[d]$,

$\Rightarrow \mu_* \mathbb{C}_M[d] \in {}^pD_c^{\geq 0}(N).$

Hence, it's perverse.

3) Similarly, we can prove 3).

Let $U \subseteq N$, s.t. $\mu|_{\mu^{-1}(U)}$ is a finite morphism.

We can shrink U st. $\mu_* \mathbb{C}_M[d]|_U \in \mathrm{Loc}(U)$. $N_0 = U$.

Let $Z = N \setminus U$, same argument as in 1) shows

$$i_Z^* \mu_* \mathbb{C}_M[d] \in {}^pD_c^{\leq -1}(Z)$$

By Verdier duality, $i_Z^! \mu_* \mathbb{C}_M[d] \in {}^pD_c^{\geq 1}(Z)$.

$\Rightarrow \mu_* \mathbb{C}_M[d] = IC(\mu_* \mathbb{C}_M[d]|_{N_0})$.

2) $x \in N_\alpha$.

$$\begin{tikzcd} M_x \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{\mu}"] & M \arrow[d, "\mu"] \\ \{x\} \arrow[r, hook, "i"] & N \end{tikzcd}$$

$$\begin{aligned} H_k(M_x) &= H^{-k}(M_x, \tilde{i}^! \mathbb{D}_M) \\ &= H^{-k}(\{x\}, \tilde{\mu}_* \tilde{i}^! \mathbb{D}_M) \\ &= H^{d-k}(\{x\}, i^! \mu_* \mathbb{C}_M[d]) \\ &= H^{d-k}(\{x\}, \bigoplus_\phi L_\phi \otimes i^! IC_\phi) \\ &= \bigoplus_\phi L_\phi \otimes H^{d-k}(\{x\}, i^! IC_\phi). \end{aligned}$$

$2d_x + n_\alpha \leq d$

if $N_\alpha \not\subseteq \overline{N_\phi}$, $i^! IC_\phi = 0$.

if $N_\alpha \subseteq \overline{N_\phi}$, $i^! IC_\phi = (i^* IC_\phi^\vee)^\vee$

Take $k = d - \dim N_\alpha \geq 2 \dim M_x$,

then $H_k(M_x) \neq 0$ only when N_α is a relevant strata.

for such a strata, ($n_\alpha = \dim N_\alpha$)

$$H^{n_\alpha}(\{x\}, i^! IC_\phi) = H^{n_\alpha}(\{x\}, (i^* IC_\phi^\vee)^\vee)$$

$$= \left(H^{-n_\alpha}(\{x\}, i^* IC(\mathcal{L}_\phi^*)) \right)^*$$

$$= \left(\mathcal{H}_x^{-n_\alpha} IC(\mathcal{L}_\phi^*) \right)^*$$

if $N_\alpha \neq N_\phi$, then $n_\alpha < \dim N_\phi$,

and $\dim \operatorname{Supp} \mathcal{H}^{-n_\alpha} IC(\mathcal{L}_\phi^*) < n_\alpha$.

Since $IC(\mathcal{L}_\phi^*)$ is locally constant on N_α,

$$\mathcal{H}_x^{-n_\alpha} IC(\mathcal{L}_\phi^*) = 0 \quad \forall x \in N_\alpha.$$

if $N_\alpha = N_\phi$, $\mathcal{H}_x^{-n_\alpha} IC(\mathcal{L}_\phi^*) = (\mathcal{L}_\phi^*)_x$

$$\Rightarrow H_{d-\dim N_\alpha}(M_x) = \bigoplus_\phi L_\phi \otimes (\mathcal{L}_\phi)_x.$$

where the sum is over all ϕ s.t. $N_\phi = N_\alpha$.

Hence, in the decomposition theorem, only the relevant pairs $\phi = (N_\phi, \chi_\phi)$ appears, and

$$L_\phi = \operatorname{Hom}_{\pi_1(N_\phi)}(\chi_\phi, H_{2\dim M_x}(M_x)) \qquad \square$$

Example: 1) $\mu: \tilde{N} \to N$ is semismall.

$N = \sqcup \mathcal{O}$, $x \in \mathcal{O}$, $\mu_Z : Z := \tilde{N} \times_N \tilde{N} \to N$

$Z_{\mathcal{O}} := \mu_Z^{-1}(\mathcal{O}) = G \times_{G_x} (\mathcal{B}_x \times \mathcal{B}_x)$.

$\Rightarrow \dim \mathcal{O} + 2\dim \mathcal{B}_x = \dim Z = \dim \tilde{N}$.

$\Rightarrow \mu$ is semismall, and every strata is relevant.

Let $\mathbb{C} \otimes_{\mathbb{Q}} H(\mathcal{B}_x) = \bigoplus_{\psi \in C(x)^\wedge} H(\mathcal{B}_x)_\psi \otimes \psi$.

where $C(x)^\wedge$:= finite dim'l reps of $C(x)$ appearing in $H(\mathcal{B}_x) \otimes_{\mathbb{Q}} \mathbb{C}$.

Hence,

$$\mu_* \mathbb{C}_{\tilde{N}}[\dim \tilde{N}]$$

$$= \bigoplus_{(\mathcal{O}, \psi)} H(\mathcal{B}_x)_\psi \otimes IC(\bar{\mathcal{O}}, \mathcal{L}_\psi).$$

2) Thm: the Grothendieck-Springer resolution

$\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is small.

Pf: $\forall n \in \mathbb{Z}$, let

$$\mathfrak{g}_n := \{ x \in \mathfrak{g} \mid \dim \mathcal{B}_x = n \}$$

Take a stratification of $\mathfrak{g}$, which is a refinement of

$$\mathfrak{g} = \mathfrak{g}_{rs} \cup (\mathfrak{g}_0 \setminus \mathfrak{g}_{rs}) \cup \bigcup_{n \geq 1} \mathfrak{g}_n$$

Over the dense strata $\mathfrak{g}_{rs}$, μ is a $|W|$ to 1 cover.

$$\dim \mathfrak{g}_{rs} = \dim \mathfrak{g} = \dim \tilde{\mathfrak{g}}$$

For any other strata $S \subseteq \mathfrak{g}_n$,

$$\mu^{-1}(S) = G \times_B (S \cap \mathfrak{b})$$

$$\begin{aligned} \dim S + 2n &= \dim \mu^{-1}(S) + n \\ &= \dim (S \cap \mathfrak{b}) + n + \dim \mathcal{B} \\ &= \dim ((S \cap \mathfrak{b}) \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) + \dim \mathcal{B} \\ &< \dim (\mathfrak{b} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) + \dim \mathcal{B} \end{aligned}$$

$$\begin{tikzcd} (S \cap \mathfrak{b}) \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{\mathfrak{g}} \arrow[d] \\ S \cap \mathfrak{b} \arrow[r, hook] & \mathfrak{g} \end{tikzcd}$$

strict $<$ since S is not the dense strata.

Thus, we only need to show $\dim \mathfrak{b}\times_{\mathfrak{g}}\tilde{\mathfrak{g}} = \dim \mathfrak{b}$.

$$\begin{array}{ccc} \mathfrak{b}\times_{\mathfrak{g}}\tilde{\mathfrak{g}} = \{ (x,\mathfrak{b}') & \mid & \mathfrak{b}'\in\mathcal{B},\ x\in\mathfrak{b}\cap\mathfrak{b}' \} \\ \downarrow\varphi & & \\ \mathcal{B} & & \end{array} \qquad \begin{array}{c} (x,\mathfrak{b}') \\ \big\downarrow \\ \mathfrak{b}' \end{array}$$

over $\mathcal{B}_w := BwB/B$, φ is

$$\{ (x, gB) \mid x\in \mathfrak{b}\cap g.\mathfrak{b},\ gB\in BwB \}$$

$$\dim(\mathfrak{b}\cap g.\mathfrak{b}) + \dim BwB = \dim\mathfrak{b}$$

Since $BwB \simeq B/B\cap g.B$

$\square$