Sheaf-theoretic analysis of the convolution alg

$M_i/\mathbb{C}$ smooth variety, $\mu_i : M_i \to N$ proper, $i=1,2$

$$\begin{tikzcd}
Z_{12} := M_1 \times_N M_2 \arrow[r, hook, "\tilde{i}"] \arrow[d, "\mu_{12}"] \arrow[dr, phantom, "\square"] & M_1 \times M_2 \arrow[d, "\mu_1 \times \mu_2"] \\
N = N_\Delta \arrow[r, hook, "i"] & N \times N
\end{tikzcd}$$

<u>Lemma</u>: $A_1 \in D^b_c(M_1)$, $A_2 \in D^b_c(M_2)$.

then $\exists$ natural isomorphism of graded vector spaces

$$H^*(Z_{12}, \tilde{i}^!(A_1^\vee \boxtimes A_2)) \simeq \mathrm{Ext}^*_{D^b_c(N)}(\mu_{1*}A_1, \mu_{2*}A_2).$$

pf: $$H^*(Z_{12}, \tilde{i}^!(A_1^\vee \boxtimes A_2)) = H^*(N, \mu_{12*}\tilde{i}^!(A_1^\vee \boxtimes A_2))$$
$$= H^*(N, i^!(\mu_{1*}A_1^\vee \boxtimes \mu_{2*}A_2)) \quad \mu_1 \text{ proper}$$
$$= H^*(N, i^!((\mu_{1*}A_1)^\vee \boxtimes \mu_{2*}A_2))$$
$$= H^*(N, \mu_{1*}A_1 \overset{!}{\otimes} \mu_{2*}A_2)$$

$$= H^*(N, \mathcal{H}om(\mu_{1*}A_1, \mu_{2*}A_2))$$

$$= \mathrm{Ext}^*_{D^b(N)}(\mu_{1*}A_1, \mu_{2*}A_2) \qquad \square$$

<u>Cor</u>: Take $A_1 = \mathbb{C}_{M_1}[m_1]$, $A_2 = \mathbb{C}_{M_2}[m_2]$, $m_i = \dim_{\mathbb{C}} M_i$

Then $H_*(Z_{12}) \simeq \mathrm{Ext}^{m_1+m_2-*}_{D^b(N)}(\mu_{1*}\mathbb{C}_{M_1}[m_1], \mu_{2*}\mathbb{C}_{M_2}[m_2])$

Pf: $\mathbb{D}_{Z_{12}} = \tilde{i}^! \mathbb{D}_{M_1 \times M_2} = \tilde{i}^!(A_1 \boxtimes A_2)[m_1+m_2]$.

$\Rightarrow H_*(Z_{12}) = H^{-*}(Z_{12}, \tilde{i}^!(A_1 \boxtimes A_2)[m_1+m_2])$

$$= \mathrm{Ext}^{m_1+m_2-*}_{D^b(N)}(\mu_{1*}\mathbb{C}_{M_1}[m_1], \mu_{2*}\mathbb{C}_{M_2}[m_2]).$$

$\square$

Now let $M_1 = M_2 = M$. $Z = M \times_N M$.

$\mathrm{Ext}^*_{D^b(N)}(\mu_*\mathbb{C}_M[m], \mu_*\mathbb{C}_M[m])$ has a product structure via the Yoneda product.

Yoneda product.

$A_1, A_2, A_3 \in D^b_c(N)$.

$$\begin{array}{ccccc}
\mathrm{Hom}_{D^b_c(N)}(A_1, A_2[p]) & \times & \mathrm{Hom}_{D^b_c(N)}(A_2[p], A_3[p+q]) & \longrightarrow & \mathrm{Hom}_{D^b_c(N)}(A_1, A_3[p+q]) \\
 & & \wr\vert & & \\
\wr\vert & & \mathrm{Hom}_{D^b(N)}(A_2, A_3[q]) & & \wr\vert \\
 & & \wr\vert & & \\
\mathrm{Ext}^p_{D^b_c(N)}(A_1, A_2) & \times & \mathrm{Ext}^q_{D^b_c(N)}(A_2, A_3) & \longrightarrow & \mathrm{Ext}^{p+q}_{D^b_c(N)}(A_1, A_3)
\end{array}$$

Hence, $\mathrm{Ext}^*_{D^b_c(N)}(A, A)$ has a product structure.

Theorem

The isomorphism $H_*(Z) \simeq \mathrm{Ext}^{2m-*}_{D^b_c(N)}(\mu_* \mathbb{C}[m], \mu_* \mathbb{C}[m])$

(not grading preserving) is an algebra isomorphism.

Sketch: Let's study the sheaf-theoretic convolution.

Setup: M_i $(i=1,2,3)$ connected manifolds of complex dim's m_i.

$\mu_i : M_i \to N$ proper. $\quad A_i \in D^b_c(M_i)$

$\varepsilon_{ij} : Z_{ij} := M_i \times_N M_j \hookrightarrow M_i \times M_j \qquad A_{ij} := \varepsilon_{ij}^! (A_i^\vee \boxtimes A_j)$

$Z_{12} \circ Z_{23} := \text{image}(Z_{12} \times_{M_2} Z_{23} \longrightarrow M_1 \times M_3) \subseteq Z_{13}.$

We will define a convolution

$$* : H^p(Z_{12}, A_{12}) \otimes H^q(Z_{23}, A_{23}) \longrightarrow H^{p+q}(Z_{13}, A_{13})$$

$$\begin{tikzcd}
Z_{12} \times_{M_2} Z_{23} \arrow[r, "\rho"] \arrow[d, hook, "\phi"] \arrow[dr, phantom, "\square"] & M_1 \times (M_2)_\Delta \times M_3 \arrow[d, hook, "\tilde{\phi}"] \\
Z_{12} \times Z_{23} \arrow[r, "h"] & M_1 \times M_2 \times M_2 \times M_3
\end{tikzcd}$$

$$H^*(Z_{12}, A_{12}) \otimes H^*(Z_{23}, A_{23}) = H^*(Z_{12} \times Z_{23}, A_{12} \boxtimes A_{23})$$

$$= H^*((\varepsilon_{12} \times \varepsilon_{23})^! (A_1^\vee \boxtimes A_2 \boxtimes A_2^\vee \boxtimes A_3))$$

$$= H^*(h^!(A_1^\vee \boxtimes A_2 \boxtimes A_2^\vee \boxtimes A_3)) \qquad h = \varepsilon_{12} \times \varepsilon_{23}$$

$$\longrightarrow H^*(\phi_* \rho^! \tilde{\phi}^* (A_1^\vee \boxtimes A_2 \boxtimes A_2^\vee \boxtimes A_3)) \qquad h^! \to h^! \tilde{\phi}_* \tilde{\phi}^* = \phi_* \rho^! \tilde{\phi}^*$$

$$= H^*(p^! \tilde{q}^*(A_1^\vee \otimes A_2 \otimes A_2^\vee \otimes A_3))$$

$$= H^*(Z_{12} \times_{M_2} Z_{23},\ p^!(A_1^\vee \otimes (A_2 \otimes A_2^\vee) \otimes A_3)) \quad \text{----} \quad (*)$$

If $A_i = \mathbb{D}_{M_i} = \mathbb{C}_{M_i}[2m_i]$,

then $\mathcal{A}_{ij} = \varepsilon_{ij}^!(\mathbb{D}_{M_i}[-2m_i] \otimes \mathbb{D}_{M_j}) = \mathbb{D}_{Z_{ij}}[-2m_i]$

Hence $H^*(Z_{ij}, \mathcal{A}_{ij}) \simeq H_*(Z_{ij})$,

and the above composition $(*)$ is nothing but the intersection pairing $H_*(Z_{12}) \otimes H_*(Z_{23}) \to H_*(Z_{12} \times_{M_2} Z_{23})$ involved in the definition of the convolution.

Recall $\mathrm{Hom}_{D^b_c(M)}(\mathbb{C}_{M_2}, A_2^\vee \overset{!}{\otimes} A_2) = H^0(M_2, \mathcal{H}om(A_2, A_2))$

$$= \mathrm{Hom}_{D^b_c(M_2)}(A_2, A_2)$$

$\rightsquigarrow \exists$ canonical morphism $\mathbb{C}_{M_2} \to A_2^\vee \overset{!}{\otimes} A_2 \rightsquigarrow A_2^\vee \otimes A_2 \to \mathbb{D}_{M_2}$

$\mathrm{Hom}(\mu_!(A_2 \otimes A_2^\vee), \mathbb{D}_N) = \mathrm{Hom}(A_2 \otimes A_2^\vee, \mathbb{D}_{M_2})$.

$\rightsquigarrow$ canonical morphism $\mu_!(A_2 \otimes A_2^\vee) \to \mathbb{D}_N$.

$$\begin{tikzcd} Z_{12}\times_{M_2} Z_{23} \arrow[r, "f"] \arrow[d, "\tilde{m}"] \arrow[dr, phantom, "\square"] & M_1\times (M_2)_\Delta \times M_3 \arrow[d, "m."] \\ N_\Delta \arrow[r, "\tilde{f}"] & N\times N_\Delta \times N \end{tikzcd}$$

$$H^*(Z_{12}\times_M Z_{23}, f^!(A_1^\vee \boxtimes (A_2\otimes A_2^\vee)\boxtimes A_3))$$

$$= H^*(N_\Delta, \tilde{f}^! m_*(A_1^\vee \boxtimes (A_2\otimes A_2^\vee)\boxtimes A_3))$$

$m_{1*} = m_{1!}$

$$= H^*(N_\Delta, \tilde{f}^!((m_{1*}A_1)^\vee \boxtimes m_{2!}(A_2\otimes A_2^\vee)\boxtimes m_{3*}A_3))$$

$m_{1*}A_1^\vee = (m_{1*}A_1)^\vee$

$$\longrightarrow H^*(N_\Delta, \tilde{f}^!((m_{1*}A_1)^\vee \boxtimes ID_{N_\Delta}\boxtimes m_{3*}A_3))$$

$$= H^*(N_\Delta, (m_{1*}A_1)^\vee \overset{!}{\otimes} ID_{N_\Delta} \overset{!}{\otimes} m_{3*}A_3)$$

$$= H^*(N_\Delta, (m_{1*}A_1)^\vee \overset{!}{\otimes} m_{3*}A_3).$$

$$= H^*(N_\Delta, \mathcal{H}om(m_{1*}A_1, m_{3*}A_3))$$

$$= \mathrm{Ext}^*(m_{1*}A_1, m_{3*}A_3) \simeq H^*(Z_{13}, A_{13}) \cdots\cdots (**)$$

The sheaf-theoretic convolution

$$H^*(Z_{12}, A_{12}) \otimes H^*(Z_{23}, A_{23}) \longrightarrow H^*(Z_{13}, A_{13}) \quad \cdots \quad (***)$$

is defined as the composition of $(*)$ and $(**)$.

Observe that if $A_i = \mathbb{D}_{M_i}$,

$$(**) = \left(H_*(Z_{12} \times_M Z_{23}) \xrightarrow[\text{direct image}]{} H_*(Z_{13}) \right)$$

Thus, $(***)$ = convolution in Borel-Moore homology.

<u>Prop</u> (8.6.35)

$$\begin{tikzcd}
H_*(Z_{12}, A_{12}) \otimes H_*(Z_{23}, A_{23}) \arrow[r, "(***)"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & H_*(Z_{13}, A_{13}) \arrow[d, equal] \\
\mathrm{Ext}^*_{D^b_c(N)}(\mu_{1*}A_1, \mu_{2*}A_2) \otimes \mathrm{Ext}^*_{D^b_c(N)}(\mu_{2*}A_2, \mu_{3*}A_3) \arrow[r, "\text{composition}"'] & \mathrm{Ext}^*_{D^b_c(N)}(\mu_{1*}A_1, \mu_{3*}A_3)
\end{tikzcd}$$

See the textbook for the proof.

$\square$

Classification of simple modules.

$\mu: M \to N$ projective, $N = \bigsqcup N_\alpha$ stratification, s.t.

$\mu: \mu^{-1}(N_p) \to N_p$ is a locally trivial fibration.

the decomposition theorem gives

$$\mu_* \mathbb{C}_M[m] = \bigoplus_{\substack{\phi = (N_p, \chi_p) \\ k \in \mathbb{Z}}} L_\phi(k) \otimes IC_\phi[k]$$

$$\Rightarrow H_*(Z) = \bigoplus_{k \in \mathbb{Z}} \mathrm{Ext}^k_{D^b_c(N)}(\mu_* \mathbb{C}_M[m], \mu_* \mathbb{C}_M[m])$$

$$= \bigoplus_{\substack{i,j,k \\ \phi,\psi}} \mathrm{Hom}_{\mathbb{C}}(L_\phi(i), L_\psi(j)) \otimes \mathrm{Ext}^k_{D^b_c(N)}(IC_\phi[i], IC_\psi[j])$$

$$= \bigoplus_{\substack{i,j,k \\ \phi,\psi}} \mathrm{Hom}_{\mathbb{C}}(L_\phi(i), L_\psi(j)) \otimes \mathrm{Ext}^k_{D^b_c(N)}(IC_\phi, IC_\psi)$$

$$L_\phi := \bigoplus_{i \in \mathbb{Z}} L_\phi(i). \qquad \mathrm{Ext}^k_{D^b_c(N)}(IC_\phi, IC_\psi) = \begin{cases} 0 & k < 0, \text{ or } k = 0, \phi \neq \psi \\ \mathbb{C} & k = 0, \phi = \psi \end{cases}$$

$$\rightsquigarrow H_*(2) = \underbrace{\bigoplus_{\phi} \operatorname{End}_{\mathbb{C}} L_\phi}_{\substack{\text{a matrix alg.} \\ \text{semisimple.}}} \oplus \underbrace{\left(\bigoplus_{\substack{\varphi, \psi \\ k>0}} \operatorname{Hom}_{\mathbb{C}}(L_\varphi, L_\psi) \otimes \operatorname{Ext}^k_{D^b_c(\mathcal{N})}(IC_\varphi, IC_\psi) \right)}_{\substack{\| \\ H_*(2)_+ \ \text{a nilpotent ideal in } H_*(2)}}$$

Since $H_*(2) / H_*(2)_+ \simeq \bigoplus_{\phi} \operatorname{End} L_\phi$ is semisimple.

$\Rightarrow H_*(2)_+ =$ Jacobson radical of $H_*(2)$.

Hence, $\forall\ \psi$,

$$H_*(2) \twoheadrightarrow H_*(2)/H_*(2)_+ = \bigoplus_{\psi} \operatorname{End} L_\psi \rightarrow \operatorname{End} L_\psi$$

yields an irreducible rep L_ψ of $H_*(2)$.

<u>Thm</u>: The non-zero members of the collection $\{L_\varphi\}$ form a complete list of the isomorphism classes of simple $H_*(2)$-modules.

Semismall case.

$\mu: M \to N$ semismall. $N = \sqcup N_\alpha$.

$(\dim M - \dim N_\alpha \geq 2 \dim \mu^{-1}(x_\alpha), \ x_\alpha \in N_\alpha)$

Decomposition theorem

$$\mu_* \mathbb{C}_M[m] = \bigoplus_{\phi = (N_\phi, \chi_\phi)} L_\phi \otimes IC_\phi ,$$

where the summation is over the relevant pairs, i.e.

$m - \dim N_\phi = 2 \dim \mu^{-1}(x)$, χ_ϕ appears as a $\pi_1(N_\phi, x)$ sub-rep of $H_{top}(\mu^{-1}(x))$, $x \in N_\phi$.

Normalized grading.

$Z = M \times_N M$.

$H_{[p]}(Z) := H_{2m-p}(Z)$.

Then $H_{[p]}(Z) * H_{[q]}(Z) \subseteq H_{[p+q]}(Z)$.

$(2m-p + 2m-q - 2m = 2m - (p+q))$.

Prop: $\exists$ graded alg. isomorphism:

$$\bigoplus_{p\geq 0} H_{[p]}(Z) \simeq \bigoplus_{p\geq 0}\left(\sum_{\phi,\psi} \mathrm{Hom}_{\mathbb{C}}(L_\phi, L_\psi) \otimes \mathrm{Ext}^p_{D^b_c(N)}(IC_\phi, IC_\psi)\right)$$

Pf: We have already proved

$$H_{-i}(Z) \simeq \mathrm{Ext}^{2m+i}_{D^b_c(N)}(\mu_*\mathbb{C}[m], \mu_*\mathbb{C}[m]).$$

$$\Rightarrow H_{[i]}(Z) \simeq \mathrm{Ext}^i_{D^b_c(N)}(\mu_*\mathbb{C}[m], \mu_*\mathbb{C}[m]).$$

$\square$

Cor: $H_{[0]}(Z) = \bigoplus_\phi \mathrm{End}\, L_\phi$ is the maximal semisimple subalg of $H_*(Z)$.

Recall $L_\phi = H_{2\dim M_x}(M_x)_\phi$, $x \in N_\phi$.

$$\Rightarrow H_{[0]}(Z) = \bigoplus_{N_\alpha,\ \chi \in \widehat{\pi_1(N_\alpha)}} \mathrm{End}\, H_{2\dim M_x}(M_x)_\phi.$$