Revisiting the Springer theory:

$\mu: \tilde{N} =: M \to N$. semismall

$$\mu_* \mathbb{C}_{\tilde{N}}[\dim \tilde{N}] = \bigoplus_{\mathcal{O}, \psi} H_{top}(\mathcal{B}_x)_\psi \otimes IC(\mathcal{O}, \psi).$$

$\psi \in \widehat{C(x)}$ = irreps of $C(x)$ appearing in $H_{top}(\mathcal{B}_x)$.

<u>Prop</u>: $H_{top}(Z) = \bigoplus_{\mathcal{O}, \psi} \mathrm{End}_{\mathbb{C}} H_{top}(\mathcal{B}_x)_\psi$.

This gives a classification of simple $H_{top}(Z)$-modules.

Combined with the Lagrangian construction of the Weyl groups

$$H_{top}(Z) \simeq \mathbb{C}[W],$$

We get all the irreducible W-mods.

We are going to give another construction of the isomorphism

$H_{top}(Z) \simeq \mathbb{C}[W]$. using sheaf theory.

<u>Goal</u>: give a new proof of $H(2) \simeq \mathbb{C}[W]$

via sheaves.

<u>Fourier transform</u>

E complex holomorphic vector bundle on X.
$\downarrow$
X complex manifold

<u>Def</u> A sheaf of $\mathbb{C}$-vector space on the total space of E is called monodromic if it's locally constant over the orbits of the natural $\mathbb{C}^*$-action on E.

A complex of sheaves is called monodromic if all its cohomology sheaves are monodromic.

$D^b_{mon}(E) :=$ full subcat of $D^b_c(E)$ consisting of monodromic complexes

$\mathrm{Perv}_{mon}(E)$ = monodromic perverse sheaves

Consider E as a real vector bundle, the complex dual bundle E^* can be identified with the real dual of E via

$(x, \xi) \mapsto \langle x, \xi \rangle := \mathrm{Re}(\xi(x))$, $x \in E$, $\xi \in E^*$.

$I^\cdot$ = a complex of injective sheaves on E, bounded below.

$\tau : E \to X$, $\check{\tau} : E^* \to X$.

$U \subseteq E^*$ open, define $U^\circ \subseteq E$ as the set of all $x \in E$, s.t.

(i) $\tau(x) \in \check{\tau}(U)$.

(ii) $\langle x, \xi \rangle \geq 0$ $\forall \xi \in U$, satisfying $\check{\tau}(\xi) = \tau(x)$.

$U^\circ \subseteq \tau^{-1}\check{\tau}(U) =: \tilde{U}$.

$U \mapsto \Gamma_{U^\circ}(\tilde{U}, I^\cdot|_{\tilde{U}})$ defines a complex of presheaves on E^*.

define $\tilde{\mathcal{F}}(I^\cdot)$ = sheafification of this complex.

Using injective resolutions of monodromic complexes,

$\rightsquigarrow \tilde{\mathcal{F}} : D^b_{mon}(E) \to D^b_{mon}(E^*)$.

$\mathcal{F} := \tilde{\mathcal{F}}[r]$, $r = \mathrm{rk} E$.

Rmk: the definition is very technical, we will only use its properties.

Prop: 1). $\mathcal{F} \circ \mathcal{F} \simeq (-1)^*$, $(-1) : E \to E$ multiplication by -1

2) $\mathcal{F} : \mathrm{Perv}_{mon}(E) \xrightarrow{\sim} \mathrm{Perv}_{mon}(E^*)$.

3). $i_V : V \hookrightarrow E$ subbundle, $i_{V^\perp} : V^\perp \hookrightarrow E^*$ annihilator of the subbundle V.

then $\mathcal{F}(i_{V*}\mathbb{C}_V[\dim V]) = (i_{V^\perp})_* \mathbb{C}_{V^\perp}[\dim V^\perp]$

Given a vector space E, $E_X := E \times X \xrightarrow{pr_E} E$

Prop: For a compact alg variety X,

$$\begin{tikzcd}
\mathrm{Perv}_{mon}(E_X) \arrow[r, "\mathcal{F}_{E_X}"] \arrow[d, "(pr_E)_*"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Perv}_{mon}(E^*_X) \arrow[d, "(pr_{E^*})_*"] \\
\mathrm{Perv}_{mon}(E) \arrow[r, "\mathcal{F}_E"] & \mathrm{Perv}_{mon}(E^*)
\end{tikzcd}$$

Fourier transform for the bundle $\begin{matrix} E \\ \downarrow \\ pt. \end{matrix}$

Back to the Springer resolution $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$.

Consider the trivial vector bundle on $\mathcal{B}$, $\mathfrak{g}_{\mathcal{B}} := \mathcal{B} \times \mathfrak{g}$
$$\downarrow$$
$$\mathcal{B}$$

$$\begin{array}{ccc} \tilde{\mathcal{N}} & \hookrightarrow & \mathfrak{g}_{\mathcal{B}} \\ \downarrow & & \downarrow \\ \mathcal{B} & = & \mathcal{B} \end{array}$$

$\tilde{\mathcal{N}} \subseteq \mathfrak{g}_{\mathcal{B}}$ subbundle.

the orthogonal subbundle is $\tilde{\mathfrak{g}}$ ($\mathfrak{g} \simeq \mathfrak{g}^*$ as $\mathfrak{g}$ is semisimple)

Hence $\mathcal{F}_{\mathfrak{g}_{\mathcal{B}}}(\mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}]) = \mathbb{C}_{\tilde{\mathfrak{g}}}[\dim \tilde{\mathfrak{g}}]$.

$$\begin{array}{c} \mathfrak{g}_{\mathcal{B}} \\ \downarrow \mathrm{pr}_{\mathfrak{g}} \\ \mathfrak{g} \end{array}$$

whose restrictions to $\tilde{\mathcal{N}}$ and $\tilde{\mathfrak{g}}$ are μ.

$$\Rightarrow \mathrm{pr}_{\mathfrak{g}*} \mathcal{F}_{\mathfrak{g}_{\mathcal{B}}}(\mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}]) = \mathcal{F}_{\mathfrak{g}}\, \mathrm{pr}_{\mathfrak{g}*} \mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}]$$

$$\Rightarrow \mu_* \mathbb{C}_{\tilde{\mathfrak{g}}}[\dim \tilde{\mathfrak{g}}] = \mathcal{F}_{\mathfrak{g}}\, \mu_* \mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}]$$

where $\mathcal{F}_{\mathfrak{g}}$ = Fourier transform on $\begin{array}{c} \mathfrak{g} \times pt \\ \downarrow \\ pt \end{array}$.

Recall $H(2) \simeq \mathrm{Ext}^{\circ}_{D^b_c(N)}(\mu_* \mathbb{C}_{\tilde{N}}[\dim \tilde{N}], \mu_* \mathbb{C}_{\tilde{N}}[\dim \tilde{N}])$

$\mathcal{F}_g$ is an equivalence $\rightarrow$ $\simeq \mathrm{Ext}^{\circ}_{D^b_c(N)}(\mathcal{F}_g \mu_* \mathbb{C}_{\tilde{N}}[\dim], \mathcal{F}_g \mu_* \mathbb{C}_{\tilde{N}}[\dim])$

$$\simeq \mathrm{End}(\mu_* \mathbb{C}_{\tilde{g}}[\dim \tilde{g}]).$$

Since $\mu : \tilde{g} \to g$ is small,

$$\mu_* \mathbb{C}_{\tilde{g}}[\dim \tilde{g}] = IC(\mu_* \mathbb{C}_{\tilde{g}}[\dim \tilde{g}]|_{g^{rs}}).$$

Hence, by the perverse continuation property.

$$H(2) \simeq \mathrm{End}(\mu_* \mathbb{C}_{\tilde{g}}[\dim \tilde{g}]|_{g^{rs}})$$

decompose the local system

$$\mu_* \mathbb{C}_{\tilde{g}}|_{g^{rs}} = \bigoplus_{\psi} L_\psi \otimes \mathcal{L}_\psi \quad \cdots\cdots (*).$$

$\mathcal{L}_\psi$ irreducible local systems on g^{rs}, L_ψ multiplicity spaces.

$$\Rightarrow H(2) \simeq \mathrm{End}(\mu_* \mathbb{C}_{\tilde{g}}[\dim \tilde{g}]|_{g^{rs}}) = \bigoplus_{\psi} \mathrm{End}\, L_\psi.$$

Recall $\mu : \tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ is a Galois cover with automorphism group W.

$\rightsquigarrow \pi_1(\mathfrak{g}^{rs}) \twoheadrightarrow W$.

Hence, looking at the stalks, the decomposition is just:

$$\mu_* \mathbb{C}_{\tilde{\mathfrak{g}}^{rs}}|_x \simeq \mathbb{C}[W] = \bigoplus_{\psi} L_\psi \otimes \mathcal{L}_\psi|_x, \qquad x \in \mathfrak{g}^{rs}.$$

and each $\mathcal{L}_\psi|_x$ is an irreducible representation of the Weyl group.

Thus, the multiplicity spaces $L_\psi \simeq$ dual vector space of the representation $\mathcal{L}_\psi|_x$.

Therefore, $\mathbb{C}[W] = \bigoplus_{\psi} \operatorname{End} L_\psi \simeq H(Z)$. $\square$

This finishes the sheaf-theoretic approaches the Springer theory for Weyl groups W.

<u>Rmks</u>: 1) Recall our conventions. $\exists$ two natural actions of W on $H^*(\mathcal{B}_e)$ that differ by tensoring with the sign rep. of W.

Our convention is that, for $x=0$, $H_{top}(\mathcal{B}_0) = H_{top}(\mathcal{B}) =$ Sign rep.; for $x \in \mathcal{N}^{reg}$, $H_{top}(\mathcal{B}_x) =$ trivial rep.

2) Springer's original construction.

$\tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ is a W-cover $\Rightarrow$ W acts on $\mu_* \mathbb{C}_{\tilde{\mathfrak{g}}}[\dim \tilde{\mathfrak{g}}]|_{\mathfrak{g}^{rs}}$.

$\xRightarrow{\text{smallness}}$ W acts on $\mu_* \mathbb{C}_{\tilde{\mathfrak{g}}}[\dim \tilde{\mathfrak{g}}] = I C(\quad)$.

$\Rightarrow$ W acts on $H^*(\mathcal{B}_x)$, $\forall x \in \mathfrak{g}$.

this differs to our convention by the Sign rep.

3). Let's assume $G = SL(n, \mathbb{C})$.

$$\mu_* \mathbb{C}_{\tilde{\mathfrak{g}}}[\dim \tilde{\mathfrak{g}}] = \bigoplus_{\lambda \vdash n} V_\lambda \otimes IC(\mathfrak{g}, \mathcal{L}_\lambda[\dim \mathfrak{g}])$$

where $\lambda \vdash n$ is a partition of n, $V_\lambda =$ the corresponding irrep. of W, $\mathcal{L}_\lambda$ the local system corresponding to $\pi_1(\mathfrak{g}^{rs}) \twoheadrightarrow S_n \to GL(V_\lambda)$.

and $V_{(n)}$ = trivial rep, $V_{(1,1,\cdots,1)}$ = sign rep.

On the other hand, we also proved

$$\mu_* \mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}] = \bigoplus_{\lambda \vdash n} H_{top}(\mathcal{B}_{x_\lambda}) \otimes IC_\lambda,$$

where $\mathcal{N} = \bigsqcup_{\lambda \vdash n} \mathcal{O}_\lambda$, IC_λ = Intersection cohomology complex on $\overline{\mathcal{O}_\lambda}$ w.r.t. to the trivial local system.

By our convention, $H_{top}(\mathcal{B}_{x_\lambda}) \simeq V_\lambda$

e.g. $\lambda = (1,1,\cdots,1)$, then $x_\lambda = 0 \in \mathcal{N}$. $H_{top}(\mathcal{B}_{x_\lambda})$ = sign rep = $V_{(1,1,\cdots,1)}$

$\lambda = (n)$, then $x_\lambda = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$, $H_{top}(\mathcal{B}_{x_\lambda})$ = trivial rep = $V_{(n)}$

$$F_{\mathfrak{g}}(\mu_* \mathbb{C}_{\tilde{\mathfrak{g}}}[\dim \tilde{\mathfrak{g}}]) = \mu_* \mathbb{C}_{\tilde{\mathcal{N}}}[\dim \tilde{\mathcal{N}}],$$

and taking into account the Weyl group actions, we need to tensor by $V_{(1,1,\cdots,1)}$ on the left hand side by remark (2), Thus

$$\bigoplus_{\lambda \vdash n} (\mathrm{sign} \otimes V_\lambda) \otimes \mathcal{F}_{\mathfrak{g}}\left(IC(\mathfrak{g}, \mathcal{L}_\lambda[\dim \tilde{\mathfrak{g}}])\right) = \bigoplus_{\lambda \vdash n} V_\lambda \otimes IC_\lambda.$$

Since $V_\lambda \otimes \mathrm{sign} = V_{\lambda^t}$, λ^t = transpose of λ,

we get

<u>Lemma</u> $\mathcal{F}\left(IC(\mathfrak{g}, \mathcal{L}_\lambda[\dim \tilde{\mathfrak{g}}])\right) = IC_{\lambda^t}$, $\forall \lambda \vdash n$.

We will need this later.

4). Some questions.

suppose $\mathfrak{b} \subseteq H \subseteq \mathfrak{sl}(n, \mathbb{C})$, H is also B-stable.

define $\mathcal{H}ess(x, H) := \{ g.B \mid Ad_{g^{-1}}(x) \in H \}$.

where $x \in \mathfrak{g}$, called Hessenberg variety.

For $x \in \mathfrak{g}^{rs}$, $\mathcal{H}ess(x, H)$ is smooth, at.

$H^*(\mathcal{H}ess(x, H))$ also carries a S_n-action.

$\forall \lambda \vdash n$. Let $M_\lambda := \mathrm{Ind}_{S_\lambda}^{S_n} \text{trivial}$, where

$$S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_l}.$$

$\{M_\lambda \mid \lambda \vdash n\}$ is a basis for $K^0(\mathrm{Rep}(S_n))$.

Thus, $H^*(\mathcal{H}ess(x, H)) = \sum_{\lambda \vdash n} c_\lambda \cdot M_\lambda$

for some $c_\lambda \in \mathbb{Z}$.

$\boxed{\text{Question: } c_\lambda \geq 0, \ \forall \lambda \vdash n.}$

Possible Strategy: put $\mathcal{H}ess(x, H)$ into a family, similar to the (Grothendieck-) Springer resolution

$$\begin{array}{ccccccc}
G \times_B \mathfrak{n} & \hookrightarrow & G \times_B \mathfrak{b} & \hookrightarrow & G \times_B H & \ni & (gB, x) \\
\downarrow & & \downarrow & & \downarrow \mu_H & & \downarrow \\
\mathcal{N} & \hookrightarrow & \mathfrak{g} & = & \mathfrak{g} & & \mathrm{Ad}_g(x).
\end{array}$$

Then $\mathcal{H}ess(x, H) = \mu_H^{-1}(x)$

Thus $H^*(\mathcal{H}ess(x,H)) \simeq (\mu_H)_* \mathbb{C}_{G\times_B H}[\dim]\big|_x$,

$$\begin{tikzcd} (\mu_H)_* \mathbb{C}_{G\times_B H}[\dim] \\ \pi_1(\mathfrak{g}^{rs}) \arrow[u, phantom, "\circlearrowright"] \arrow[r, two heads] & W \arrow[r] & 1 \end{tikzcd}$$

and the S_n-action on LHS = monodromy action on the RHS, which factors through $W = S_n$.

Can we go to the dual side and compute this?

$$\begin{tikzcd} G\times_B H^\perp \arrow[r, hook] \arrow[d, "\mu_{H^\perp}"] & G\times_B H \arrow[r, hook] \arrow[d, "\mu_H"] & \mathcal{B}\times\mathfrak{g} \\ \overline{\mathcal{U}} \arrow[r, hook] & \mathfrak{g} \end{tikzcd}$$

Answer: this is possible if $H = P_\lambda$ for some parabolic subalg,

and this is done by Borho - MacPherson.

Q: How about other H's ?