Springer theory for $\mathcal{U}(\mathfrak{sl}_n)$

1. Geometric Construction of $\mathcal{U}(\mathfrak{sl}_n)$

Fix $n \geq 1$, $d \geq 1$. $G = SL(n, \mathbb{C})$.

$$\mathcal{F} := \{ F = (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \mathbb{C}^d) \}$$

a n-step partial flag F in $\mathbb{C}^d$.

Connected components of $\mathcal{F}$ are indexed by partitions of d.

$$\underline{d} = (d_1, d_2, \cdots, d_n), \quad \sum d_i = d, \quad d_i \geq 0$$

$$\mathcal{F}_{\underline{d}} := \{ F \subseteq \mathcal{F} \mid \dim F_i / F_{i-1} = d_i \}$$

$$N := \{ x : \mathbb{C}^d \to \mathbb{C}^d \mid x \text{ linear}, \ x^n = 0 \}.$$

$$M := \{ (x, F) \in N \times \mathcal{F} \mid x(F_i) \subseteq F_{i-1}, \ \forall i = 1, 2, \cdots, n \}.$$

$$\begin{array}{ccc} M & & \\ \mu \downarrow & \searrow \pi & \\ N & & \mathcal{F} \end{array} \qquad Z := M \times_N M \subseteq T^*\mathcal{F} \times T^*\mathcal{F}.$$

Then $\mathcal{M} = \bigsqcup_{\underline{d}} \mathcal{M}_{\underline{d}}$, $\mathcal{M}_{\underline{d}} = \{(x, F) \mid F \in \mathcal{F}_{\underline{d}}\}$.

Prop: 1) $\mathcal{F}_{\underline{d}} \simeq SL(d, \mathbb{C}) / P_{\underline{d}}$, $P_{\underline{d}} = \left\{ \begin{pmatrix} \boxed{\times}_{d_1} & & \times \\ & \boxed{\times}_{d_2} \ddots & \times \\ & & \boxed{\times}_{d_n} \end{pmatrix} \right\}$

2) $\mathcal{M} \simeq T^* \mathcal{F}$, $\mathcal{M}_{\underline{d}} = T^* \mathcal{F}_{\underline{d}}$

3) (Spaltenstein) $\forall x \in \mathcal{N}$, let $\mathcal{F}_x := \pi^{-1}(x)$. then

$\mathcal{F}_x \cap \mathcal{M}_{\underline{d}}$ is connected, and of pure dimension

$\dim \mathcal{O}_x + 2 \dim \mathcal{F}_x \cap \mathcal{M}_{\underline{d}} = 2 \cdot \dim \mathcal{F}_{\underline{d}}$

4). $\#\, GL_d(\mathbb{C})$-diagonal orbits on $\mathcal{F} \times \mathcal{F} < \infty$.

5). $Z = \bigsqcup$ conormal bundle to all $GL_d(\mathbb{C})$-orbits in $\mathcal{F} \times \mathcal{F}$.

In particular, if Z^α is an irreducible component of Z

contained in $\mathcal{M}_{\underline{d}_1} \times \mathcal{M}_{\underline{d}_2}$, then $\dim Z^\alpha = \frac{1}{2} \dim (\mathcal{M}_{\underline{d}_1} \times \mathcal{M}_{\underline{d}_2})$.

Let $H(Z)$ (resp $H(\mathcal{F}_x)$)

$:=$ subspace of $H_*(Z)$ (resp $H_*(\mathcal{F}_x)$) spanned by all the

fundamental classes of irreducible components of Z. (resp $\mathcal{F}_x$).

Rmks: $H(Z) \neq H_{top}(Z)$ as Z is not of pure dimension.

By degree counting, we have

Lemma: 1) $H(Z)$ is a subalg of the convolution alg $H_*(Z)$

2) $H(\mathcal{F}_x)$ is a $H(Z)$-stable subspace of $H_*(\mathcal{F}_x)$

pf: Recall $H_i(Z_{12}) * H_j(Z_{23}) \to H_{i+j-2\dim_{\mathbb{C}} M_2}(Z_{13})$.

1). follows from the dimension formula for $Z^\alpha \subseteq Z$

2) Suppose $Z^\alpha \subseteq M_{\underline{d}_1} \times M_{\underline{d}_2}$, $c \in H(\mathcal{F}_x \cap M_{\underline{d}_2})$

then $[Z^\alpha] * c \in H_{2\dim_{\mathbb{C}} Z^\alpha + 2\dim_{\mathbb{C}}(\mathcal{F}_x \cap M_{\underline{d}_2}) - 2\dim_{\mathbb{C}} M_2}(\mathcal{F}_x \cap M_{\underline{d}_1})$

$$2\dim_{\mathbb{C}} Z^{\alpha} + 2\dim_{\mathbb{C}}(F_x \cap M_{\underline{d_2}}) - 2\dim_{\mathbb{C}} M_2$$

$$= \dim_{\mathbb{C}} M_1 - \dim_{\mathbb{C}} \mathcal{O}_x$$

$$= 2\dim_{\mathbb{C}}(F_x \cap M_{\underline{d_1}}). \qquad \square$$

<u>Thm</u>: $\exists$ a natural surjective alg. homomorphism

$$\mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \twoheadrightarrow H(Z).$$

<u>Remark</u>: $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$ is ∞-dim'l., $H(Z)$ is finite dim'l.

<u>Construction</u>

$S = \{e_\alpha, f_\alpha, h_\alpha \mid 1 \le \alpha \le n-1\}$ chevalley generators for $\mathfrak{sl}_n(\mathbb{C})$

i.e. $e_\alpha = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$, $f_\alpha = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & 1 & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$, $h_\alpha = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$

Let's first construct a map

$$\Theta : S \to H(Z)$$

$$h_\alpha \mapsto \sum_{\underline{d}} (d_\alpha - d_{\alpha+1}) \cdot [T^*_\Delta(\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}})],$$

where $\Delta \hookrightarrow \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}}$ is the diagonal embedding.

$\forall \underline{d}$, and α, let

$$\underline{d}^+_\alpha := (d_1, \cdots, d_\alpha + 1, d_{\alpha+1} - 1, d_{\alpha+2}, \cdots, d_n)$$

$$\underline{d}^-_\alpha := (d_1, \cdots, d_\alpha - 1, d_{\alpha+1} + 1, d_{\alpha+2}, \cdots, d_n)$$

Let $$Y_{\underline{d}^+_\alpha, \underline{d}} := \left\{ (F, F') \in \mathcal{F}_{\underline{d}^+_\alpha} \times \mathcal{F}_{\underline{d}} \;\middle|\; \begin{matrix} F_i = F_i', \ i \neq \alpha \\ F_\alpha \supseteq F_\alpha', \ \dim F_\alpha / F_\alpha' = 1 \end{matrix} \right\}$$

$$Y_{\underline{d}^-_\alpha, \underline{d}} = \left\{ (F, F') \in \mathcal{F}_{\underline{d}^-_\alpha} \times \mathcal{F}_{\underline{d}} \;\middle|\; \begin{matrix} F_i = F_i', \ i \neq \alpha \\ F_\alpha \subseteq F_\alpha', \ \dim F_\alpha' / F_\alpha = 1 \end{matrix} \right\}$$

Then $Y_{\underline{d}^\pm_\alpha, \underline{d}}$ is a single $GL(d, \mathbb{C})$-orbit in $\mathcal{F}_{\underline{d}^\pm_\alpha} \times \mathcal{F}_{\underline{d}}$,

and they are orbits of minimal dimension. Hence, they are smooth closed subvarieties.

Observe that $(\underline{d}^+_\alpha)^-_\alpha = \underline{d}$, $Y_{\underline{d}^+_\alpha, \underline{d}}$ and $Y_{\underline{d}, \underline{d}^+_\alpha}$ are related by

Switching the factors $\mathcal{F}_{\underline{d}_\alpha^+} \times \mathcal{F}_{\underline{d}}$.

$$\Theta(e_\alpha) := \sum_{\underline{d}} (-1)^{d_\alpha} \cdot \left[T^*_{Y_{\underline{d}_\alpha^+, \underline{d}}} (\mathcal{F}_{\underline{d}_\alpha^+} \times \mathcal{F}_{\underline{d}}) \right]$$

$$\Theta(f_\alpha) := \sum_{\underline{d}} (-1)^{d_\alpha + 1} \left[T^*_{Y_{\underline{d}_\alpha^-, \underline{d}}} (\mathcal{F}_{\underline{d}_\alpha^-} \times \mathcal{F}_{\underline{d}}) \right].$$

So $\Theta(f_\alpha) =$ transpose of $\Theta(e_\alpha)$.

We will prove later that Θ defines a surjective alg map

$$\Theta : \mathcal{U}(sl_n(\mathbb{C})) \twoheadrightarrow H(2).$$

Then either the geometric analysis of the $H(2)$-module or the sheaf-method, we get

<u>Thm</u>: $\{ H(\mathcal{F}_x) \mid x \in \mathbb{O} = G.x \subseteq N \}$ is a complete list of the isomorphism classes of simple $H(2)$-modules.

Pf: Geometric analysis needs

① $H(\mathbb{Z})$ semisimple (holds since $U(\mathfrak{sl}_2(\mathbb{C})) \twoheadrightarrow H(\mathbb{Z})$)

② $H(\mathcal{F}_x)_R \simeq (H(\mathcal{F}_x)_L)^\vee$

follows from the Cartan anti-involution on $U(\mathfrak{sl}_2(\mathbb{C}))$

$e_\alpha \Leftrightarrow f_\alpha \quad h_\alpha \leftrightarrow h_\alpha.$

and the fact that

Cartan anti-involution = switching factors of Z.

$\square$

2). finite dim'l simple $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C}))$-modules

Let $x \in \mathcal{N}$, define

$F^{max}(x) := (0 = \ker x^0 \subseteq \ker x \subseteq \ker x^2 \subseteq \cdots \subseteq \ker x^n = \mathbb{C}^d)$

$F^{min}(x) := (0 = \operatorname{Im} x^n \subseteq \operatorname{Im} x^{n-1} \subseteq \cdots \subseteq \operatorname{Im} x^0 = \mathbb{C}^d)$

then $F^{max}(x), F^{min}(x) \in \mathcal{F}_x$.

$\forall F = (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \mathbb{C}^d) \in \mathcal{F}_x$,

then $\operatorname{Im} x^{n-i} \subseteq F_i \subseteq \ker x^i$

This explains max and min.

Let $d_i(x) := \dim \ker x^i - \dim \ker x^{i-1}$.

then $(d_1(x), \cdots, d_n(x)) =: \underline{d}(x)$ is a partition of d, and $F^{max}(x) \in (\mathcal{F}_{\underline{d}})_x$.

Lemma: $\underline{d}(x)$ is a dominant $\mathfrak{gl}_n(\mathbb{C})$-weight, i.e. $d_i \geq d_{i+1}$

pf: $\ker x^{i+1} / \ker x^i \xrightarrow{x} \ker x^i / \ker x^{i-1}$ $\square$

Recall $\mathcal{U}(\mathfrak{sl}_n(\mathbb{C})) \twoheadrightarrow H(Z)$, and $\{H(\mathcal{F}_x) \mid x \in N\}/\text{conj}$ is a complete list of simple $H(Z)$-modules.

<u>Thm</u>: a) $\forall\, x \in N$, the simple $\mathfrak{sl}_n(\mathbb{C})$-module $H(\mathcal{F}_x)$ has highest weight $\underline{d}(x)$.

b) $F^{max}(x)$ (resp. $F^{min}(x)$) is an isolated point of $\mathcal{F}_x$, and $[F^{max}(x)]$ (resp. $[F^{min}(x)]$) is a highest (resp. lowest) weight vector in $H(\mathcal{F}_x)$.

Pf: Firstly, $[T_\Delta^*(\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}})]$ acts as the Id on $H((\mathcal{F}_{\underline{d}})_x)$, and 0 on $H((\mathcal{F}_{\underline{d}'})_x)$ if $\underline{d}' \neq \underline{d}$.

Recall $\Theta(h_\alpha) := \sum_{\underline{d}} (d_\alpha - d_{\alpha+1}) [T_\Delta^*(\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}})]$.

Hence, $H((\mathcal{F}_{\underline{d}})_x)$ has weights $(d_1, d_2, \dots, d_n)$.

Since $\forall\, F = (F_i) \in (\mathcal{F}_{\underline{d}})_x$, $\quad F_i \subseteq \operatorname{Ker} x^i$,

$$\Rightarrow d_1 + \cdots + d_i \leq d_1(x) + d_2(x) + \cdots + d_i(x)$$

$$\Rightarrow \underline{d}(x) \geq (d_1, d_2, \cdots, d_n)$$

Thus, $\underline{d}(x)$ is the highest weight in $H(\mathcal{F}_x)$. $\square$

<u>Remark</u>: 1) $x \in N$, $x^n = 0$. $\quad d_i(x) := \operatorname{Ker} x^i - \operatorname{Ker} x^{i-1}$.

$\underline{d}(x)$ a partition of d with at most n-rows.

the transpose $(\underline{d}(x))^t$ = sizes of the Jordan blocks of x.

e.g. $x = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}$, $\quad \underline{d}(x) = \ydiagram{2,1,1} = (2,1,1)$

$d = 4$, $n = 3$.
$$(\underline{d}(x))^t = \ydiagram{3,1} = (3,1)$$

So if $x \in \mathcal{O}_\lambda \subseteq N$, $\lambda \vdash d$, then

$H(\mathcal{F}_x)$ has highest weight λ^t.

2). simple modules of $\mathcal{U}(\mathfrak{sl}_n)$ may arise from a rep. of $H(2)$ iff the highest weight $\underline{m} = (m_1, m_2, \dots, m_n)$ is a partition of d. Such representation are precisely the simple $\mathfrak{sl}_n(\mathbb{C})$ modules that occur with non-zero multiplicity in $(\mathbb{C}^n)^{\otimes d}$

$$\underset{\mathfrak{sl}_n \curvearrowright}{(\mathbb{C}^n)^{\otimes d}} \underset{\curvearrowleft S_d}{} = \bigoplus_{\substack{\underline{m} \text{ partition} \\ \text{of } d \text{ with at} \\ \text{most } n \text{ parts}}} \underset{\substack{\cap \\ \mathrm{Irr}(\mathcal{U}(\mathfrak{sl}_n))}}{V_{\underline{m}}} \otimes \underset{\substack{\cap \\ \mathrm{Irrep}(S_d)}}{W_{\underline{m}}} \qquad \text{(Schur-Weyl)}$$

Hence, if $I_d := \mathrm{Ann}_{\mathcal{U}(\mathfrak{sl}_n)}((\mathbb{C}^n)^{\otimes d}) \subseteq \mathcal{U}(\mathfrak{sl}_n)$, we get

$$\mathcal{U}(\mathfrak{sl}_n)/I_d \xrightarrow{\sim} H(2) \simeq \bigoplus_{x/\sim} \mathrm{End}(H(\mathcal{F}_x))$$

3) generalizations in the literature.

a) Consider the $GL_d(\mathbb{C}) \times \mathbb{C}^*$-equiv. case, we get the Yangians. (Varagnolo.)

b) consider partial flags in other classical types, we get symmetric pairs.

See. works of Weiqiang Wang, Yiqiang Li and their collaborators.

c) regard $T^*\mathcal{F}_{\underline{d}}$ as a quiver variety for the type A Dynkin quiver, Nakajima considered other quivers, and constructed the corresponding reps.

4) $\exists$ other constructions of $\mathcal{U}(\mathfrak{sl}_n)$ reps via geometric Satake equivalence (the two constructions are related by symplectic duality?)