A convolution formula.

For computations, we need the following explicit convolution formula.

X_1, X_2, X_3 complex manifolds, $Y_{12} \subseteq X_1 \times X_2$, $Y_{23} \subseteq X_2 \times X_3$

$p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$.

$Pr_{ij} : T^*(X_1 \times X_2 \times X_3) \to T^*(X_i \times X_j)$

Put $Y_{13} := Y_{12} \circ Y_{23}$, $Z_{ij} = T^*_{Y_{ij}}(X_i \times X_j)$.

Thm (Thm 2.7.26 in [CG], up to a sign)

Assume: a) $p_{12}^{-1}(Y_{12})$ and $p_{23}^{-1}(Y_{23})$ intersect transversally

b) $p_{13} : p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \to Y_{13}$ is a smooth locally trivial oriented fibration with smooth base Y_{13} and smooth, compact fiber F.

Then. (i) $Z_{12} \circ Z_{23} = Z_{13}$

missing in [CG].

(ii) $[Z_{12}] * [Z_{23}] = (-1)^{\dim F} \cdot \chi(F) \cdot [Z_{13}]$,

where $\chi(F)$ = Euler characteristic of F.

Rmk: $X_1 = X_3 = pt$, X_2 compact,

$Y_{12} = pt \times X_2$, $Y_{23} = X_2 \times pt$,

Thm says $[X_2] * [X_2] = (-1)^{\dim X_2} \chi(X_2)$.

intersection in T^*X_2

Sketch of the proof:

Lemma (access intersection formula)

$Z_1, Z_2 \subseteq M$ closed, all smooth, $Z_1 \cap Z_2$ smooth.

Assume $Z_1 \cap Z_2$ is clean, i.e. $T_x Z_1 \cap T_x Z_2 = T_x Z$, $\forall x \in Z_1 \cap Z_2$

Then $[Z_1] \cap [Z_2] = e\left(T_{Z_1 \cap Z_2} M \big/ T_{Z_1 \cap Z_2} Z_1 + T_{Z_1 \cap Z_2} Z_2\right) \cdot [Z_1 \cap Z_2]$.

Euler class. = top Chern class.

$$\begin{tikzcd} Z := pr_{12}^{-1}(Z_{12}) \cap pr_{23}^{-1}(Z_{23}) \arrow[r, "\pi"] \arrow[d, "pr_{13}"] \arrow[dr, phantom, "\square"] & y := p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \arrow[d, "p_y"] \\ Z_{13} \arrow[r, "\bar{\pi}"] & Y_{13} \end{tikzcd}$$

$p_{12}^{-1}(Y_{12})$ and $p_{23}^{-1}(Y_{23})$ intersect transversally

$\Rightarrow$ $pr_{12}^{-1}(Z_{12}) \cap pr_{23}^{-1}(Z_{23})$ is clean.

Lemma $\Rightarrow$

$$[pr_{12}^{-1}(Z_{12})] \cap [pr_{23}^{-1}(Z_{23})] = e(T/T_1+T_2) \cdot [Z].$$

Where T_1, T_2 and T are normal bundle at Z to $pr_{12}^{-1}(Z_{12})$, $pr_{23}^{-1}(Z_{23})$ and $T^*(X_1 \times X_2 \times X_3)$, respectively.

then $T/(T_1+T_2) \simeq \pi^* T^*_{y/Y_{13}}$ ← relative cotangent bundle.

(In [CG], the r.h.s. is $\pi^* T_{y/Y_{13}}$, that's why the $(-1)^{\dim F}$ is missing. Let's check $X_1 = X_3 = pt$, $Y_{12} = pt \times X_2$, $Y_{23} = X_2 \times pt$,

$$
\begin{tikzcd}
Z = pt \times T^*_{X_2}X_2 \times pt \arrow[r, "\pi"] \arrow[d] & Y = pt \times X_2 \times pt \arrow[d] \\
Z_{13} = pt \times pt \arrow[r] & Y_{13} = pt \times pt
\end{tikzcd}
$$

$$
\left(T/(T_1+T_2) = T_{T^*_{X_2}X_2}(T^*X_2) \Big/ {0+0} = \pi^* T^*_{Y/Y_{13}} \right).
$$

$$
\begin{aligned}
\Rightarrow [Z_{12}] * [Z_{23}]
&= (pr_{13})_* \left(e(\pi^* T^*_{Y/Y_{13}}) \cdot [Z] \right). \\
&= (pr_{13})_* \pi^* \left(e(T^*_{Y/Y_{13}}) \cdot [Y] \right) \\
&= \bar{\pi}^* \cdot p_{Y*} \left(e(T^*_{Y/Y_{13}}) \cdot [Y] \right) \\
&= \bar{\pi}^* \left((-1)^{\dim F} \cdot \chi(F) \cdot [Y_{13}] \right) \\
&= (-1)^{\dim F} \cdot \chi(F) \cdot [Z_{13}].
\end{aligned}
$$

p_Y: locally trivial fibration with fiber F.

$\square$

<u>Example</u>: Springer theory for $\mathfrak{sl}_2$.

The flag variety $\mathcal{F} = \coprod_{0\le k\le d} Gr_k^d$ ← Grassmannian of k-planes in $\mathbb{C}^d$.

$x \in \operatorname{End}\mathbb{C}^d$, $x^2 = 0$.

$$x = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \begin{matrix} 0 & 1 \\ & 0 \end{matrix} & & \\ & & & & \ddots & \\ & & & & & \begin{matrix} 0 & 1 \\ & 0 \end{matrix} \end{pmatrix}_{d\times d}$$

k one-by-one block

l two-by-two block.

$k+2l=d$

$\underline{d}(x) = (\dim\ker x,\ d-\dim\ker x) = (k+l,\ l)$

$\Rightarrow H(\mathcal{F}_x)$ has highest weight $k+l-l=k$, which doesn't depend on l.

Assume $l=0$, then $x=0$.

$$\mathcal{F}_x = \coprod_{0\le k\le d} Gr_k^d.$$

$H(\mathcal{F}_x) = \{[Gr_k^d] \mid 0\le k\le d\}$ has dimension $d+1$.

Let's write down the action of e and f on $[Gr_k^d]$.

$0 \le i \le d$.

$$Y_i^+ := \{(F, F') \in Gr_{i+1}^d \times Gr_i^d \mid F \supseteq F',\ \dim F/F' = 1\}$$

$$Y_i^- := \{(F, F') \in Gr_{i-1}^d \times Gr_i^d \mid F \subseteq F',\ \dim F'/F = 1\}$$

$$e := \sum_i (-1)^i [T^*_{Y_i^+}(Gr_{i+1}^d \times Gr_i^d)]$$

$$\quad \| \quad T^*_{Y_i^+}$$

$$f := \sum_i (-1)^{d-i} \cdot [T^*_{Y_i^-}(Gr_{i-1}^d \times Gr_i^d)]$$

$$\quad \| \quad T^*_{Y_i^-}$$

Let's compute

$$e \cdot [Gr_k^d].$$

We use the above convolution formula.

$$M_1 = T^* Gr_{k+1}^d,\quad M_2 = T^* Gr_k^d,\quad M_3 = pt.$$

$$Z_{12} = T^*_{Y_k^+},\quad Z_{23} = Gr_k^d \times pt.$$

$X_1 = Gr^d_{k+1}, \quad X_2 = Gr^d_k, \quad X_3 = pt.$

$Y_{12} = Y^+_k, \quad Y_{23} = Gr^d_k \times pt$

$p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) = Y^+_k, \quad Y_{12} \circ Y_{23} = Gr^d_{k+1}.$

$p_{13} : Y^+_k \longrightarrow Gr^d_{k+1} \qquad \text{fiber} \simeq Gr^{k+1}_k \simeq \mathbb{P}^k$

$\Rightarrow (-1)^k [T^*_{Y^+_k}] * [Gr^d_k] = \chi(\mathbb{P}^k) \cdot [Gr^d_{k+1}] = (k+1) \cdot [Gr^d_{k+1}].$

$\Rightarrow e \cdot [Gr^d_k] = (k+1) \cdot [Gr^d_{k+1}].$

Exercise: $f \cdot [Gr^d_k] = (d-k+1) \cdot [Gr^d_{k-1}].$

Rmk:

$$[Gr^d_0] \underset{f}{\overset{e}{\rightleftarrows}} [Gr^d_1] \underset{e}{\rightleftarrows} \cdots \qquad [Gr^d_{d-1}] \underset{f}{\overset{e}{\rightleftarrows}} [Gr^d_d]$$

highest weight vector. ($\downarrow [Gr^d_d]$)

highest weight $= d$. $\quad sl_2(\mathbb{C})$ rep. $V_d \simeq \mathbb{C}[x,y]^d =$ deg d homogeneous polys.

<u>Exercise</u>: 1) do the computation for $x = \begin{pmatrix} 0 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & 0 & 1 & & & \\ & & & 0 & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & 1 \\ & & & & & & 0 & 0 \end{pmatrix}$

x has k (resp l) Jordan blocks of size 1 (resp 2).

(i.e. let's not assume $l = 0$ in the previous computation).

then $\forall F \in \pi^{-1}(x)$

$$\mathrm{Im}\, x \subseteq F \subseteq \ker x, \quad \text{thus } \pi^{-1}(x) \simeq \bigsqcup_{0 \le i \le k} Gr_i^k.$$

and $H(\pi^{-1}(x)) \simeq V_k$ = highest weight k-rep. of $\mathcal{U}(sl_2)$.

$$[Gr_0^k] \underset{f}{\overset{e}{\rightleftarrows}} [Gr_1^k] \underset{f}{\overset{e}{\rightleftarrows}} \cdots \underset{f}{\overset{e}{\rightleftarrows}} [Gr_{k-1}^k] \underset{f}{\overset{e}{\rightleftarrows}} [Gr_k^k]$$

2) use this to check the surjectivity as follows.

$$\begin{array}{c} T^*\mathcal{F} = \bigsqcup_{0 \le k \le d} T^* Gr_k^d \\ \downarrow \\ \mathcal{N} = \bigsqcup_{0 \le l \le [\frac{d}{2}]} \mathcal{N}_l \end{array}$$

$$\mathcal{N}_l = \left\{ x \in Mat_d(\mathbb{C}) \;\middle|\; x^2 = 0,\ x \sim \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 0 & 0 & \\ & & & & \ddots \end{pmatrix} \right\}$$

k blocks of size 1

l blocks of size 2

$$Z = T^*\mathcal{F} \times_{\mathcal{N}} T^*\mathcal{F} \qquad Z_{\leq l} := \bigsqcup_{j \leq l} \mu_2^{-1}(\mathcal{N}_j)$$

$$\downarrow \mu_2$$

$$\mathcal{N} \qquad Z_{<l} := \bigsqcup_{j<l} \mu_2^{-1}(\mathcal{N}_j)$$

Then $H(Z) \simeq \bigoplus_l H(Z_{\leq l})/H(Z_{<l})$ Semisimplicity

$$\simeq \bigoplus_l H(Z_l)$$

$$= \bigoplus_l H(\mathcal{F}_x)_L \otimes H(\mathcal{F}_x)_R \qquad x \in \mathcal{N}_l$$

$$\simeq V_{d-2l} \otimes V_{d-2l}^{\vee} \qquad \text{by exercise 1).}$$

<u>check</u> $\sum_{k,k'} C E^k \cdot [\Delta_{T^*Gr_l^d}] \cdot F^{k'} \simeq V_{d-2l} \otimes V_{d-2l}^{\vee}$

mod $H(Z_{\leq l-1})$.

$$\Rightarrow U(\mathfrak{sl}_2) \twoheadrightarrow H(Z)$$

Let's check the relations.

$[h,e]=2e,\quad [h,f]=2f,\quad [e,f]=h.$

let's check $[e,f]=h$.

e.f. $\quad p_{12}^{-1}(Y_{i-1}^{+})\cap p_{23}^{-1}(Y_i^{-})$

$$=\{F_1 \overset{1}{\supseteq} F_2 \overset{1}{\subseteq} F_3\}\subseteq Gr_i^d\times Gr_{i-1}^d\times Gr_i^d$$

Let $U:=Gr_i^d\times Gr_{i-1}^d\times Gr_i^d \setminus p_{13}^{-1}(\Delta)$

then $\quad p_{12}^{-1}(Y_{i-1}^{+})\cap p_{23}^{-1}(Y_i^{-})\big|_U$

$$\downarrow p_{13}$$

$Gr_i^d\times Gr_i^d\setminus\Delta$ $\quad$ is an isomorphism over it's image A.

since $F_2=F_1\cap F_3$

$(\ \dim F_1-1=\dim F_2\leq \dim F_1\cap F_3\leq \dim F_1-1)$

$$\Rightarrow [T^*_{Y_{i-1}^{+}}]*[T^*_{Y_i^{-}}]\Big|_{T^*(Gr_i^d\times Gr_i^d)\setminus\Delta} = [T^*_A(Gr_i^d\times Gr_i^d)]$$

Same for f.e.

$(F_1\overset{1}{\subseteq}F_2\overset{1}{\supseteq}F_3)$, $\dim F_1\cup F_3=\dim F_1 \Leftrightarrow \dim F_1\cap F_3=\dim F_1-1$

Thus

$$[T^*_{Y^+_{i+1}}] * [T^*_{Y^-_i}] - [T^*_{Y^-_{i+1}}] * [T^*_{Y^+_i}] = a \cdot [T^*_{\Delta}(Gr^d_i \times Gr^d_i)]$$

To compute a, we can apply the above to $[Gr^d_i]$, we get

$$a = i(d-i+1) - (d-i)(i+1)$$

$$= i - (d-i)$$

$\Rightarrow [e, f] = h.$

pf of the theorem for $\mathcal{U}(sl_n)$ $\quad\quad$ $\underline{\theta : \mathcal{U}(sl_n) \twoheadrightarrow H(Z)}$

Recall $S = \{e_\alpha, f_\alpha, h_\alpha \mid \alpha \text{ simple roots}\}$ is the Chevalley generators of $\mathfrak{g}$. $C = (C_{\alpha,\beta})$ Cartan matrix.

the Lie alg $\mathfrak{g}$ is generated by S, with relations

(1) $[h_\alpha, h_\beta] = 0$

(2) $[h_\alpha, e_\beta] = C_{\alpha,\beta} e_\beta$

(3) $[h_\alpha, f_\beta] = -C_{\alpha,\beta} e_\beta$

(4) $[e_\alpha, f_\beta] = \delta_{\alpha,\beta} h_\alpha$

(5) $(\operatorname{ad} e_\alpha)^{-C_{\alpha,\beta}+1}(e_\beta) = 0 \quad\quad \alpha \neq \beta$

(6) $(\operatorname{ad} f_\alpha)^{-C_{\alpha,\beta}+1}(f_\beta) = 0. \quad\quad \alpha \neq \beta$

We need to check these relations for $\theta(e_\alpha)$, $\theta(f_\alpha)$, $\theta(h_\alpha)$.

<u>Lemma</u>: Let the sl_2-triple (e, h, f) act on a finite dim'l vector space V, assume $v \in V$ satisfies $f \cdot v = 0$, $h \cdot v = -mv$.

Then m is a non-negative interger, and $e^{m+1} \cdot v = 0$.

pf: for an irr. sl_2 rep V, highest weight $= k \geq 0$

$$(V_k)_{-k} \underset{f}{\overset{e}{\rightleftarrows}} (V_k)_{2-k} \underset{f}{\overset{e}{\rightleftarrows}} \cdots \quad (V_k)_{k-4} \underset{f}{\overset{e}{\rightleftarrows}} (V_k)_{k-2} \underset{f}{\overset{e}{\rightleftarrows}} (V_k)_k$$

Lemma follows from this $\square$

Lemma: If a finite dim'l alg A contains $\{e_\alpha, f_\alpha, h_\alpha, \alpha \in \Delta\}$ s.t. the relations (1) - (4) holds, then (5) and (6) holds.

pf: Apply the above lemma to the adjoint rep of the sl_2-triple $(e_\alpha, f_\alpha, f_\alpha)$ to A. $\square$

Thus we only need to show

Prop: $\{\Theta(e_\alpha), \Theta(f_\alpha), \Theta(h_\alpha) \mid \alpha \in \Delta\}$ satisfy the relations (1), (2), (3) and (4).

pf: Since $[T_\Delta^*(F_{\underline{d}} \times F_{\underline{d}})]$ is the identity operator,

(1), (2), (3) would follow from this.

For (4), $[\Theta(e_\alpha), \Theta(f_\beta)] = \delta_{\alpha,\beta}\, \Theta(h_\alpha)$

easy for $\alpha \neq \beta$.

for $\alpha = \beta$, this is the same as in the sl_2 case. $\square$

Rmk: Surjectivity of Θ is proved by induction; see

[CG, Section 4.3].