§ Sheaf-theoretic approach to Springer theory for $\mathcal{U}(\mathfrak{sl}_n)$. Braverman-Gaitsgory.

Recall the notations

$$\mathcal{P}_n^d = \{ \underline{d} = (d_1, \dots, d_n) \mid \sum d_i = d \}$$

$${}^n\mathcal{F} = \{ 0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \mathbb{C}^d \} = \coprod_{\underline{d} \in \mathcal{P}_n^d} {}^n\mathcal{F}_{\underline{d}}$$

$$\mathcal{N}_n = \{ x \in \mathfrak{gl}(d,\mathbb{C}) \mid x^n = 0 \}$$

$$\begin{array}{c} {}^n\mathcal{M} = T^*({}^n\mathcal{F}) \\ \downarrow \mu_n \\ \mathcal{N}_n \end{array}$$

Let $T \subseteq GL(n,\mathbb{C})$ the diagonal torus. We work T-equivariantly.

Let $E := \mathbb{C}^n$. and $E_i = \mathbb{C}\langle e_i \rangle$, $\{e_i\}_{i=1}^n$ standard basis.

For $\underline{d} \in \mathcal{P}_n^d$, $E^{\underline{d}} := E_1^{\otimes d_1} \otimes \cdots \otimes E_n^{\otimes d_n}$.

Let ${}^n\mathcal{L}_E$ be the perverse sheaf on ${}^n\mathcal{M}$ described as follows.

for $\underline{d} \in \mathcal{P}_n^d$, ${}^n\mathcal{L}_E|_{{}^n\mathcal{N}_{\underline{d}}} = E_{{}^n\mathcal{N}_{\underline{d}}}^{\underline{d}}[2\dim {}^n\mathcal{F}_{\underline{d}}]$,

the shifted constant sheaf with 1-dim'l fiber $E^{\underline{d}}$.

Endow ${}^n\mathcal{L}_E$ with a T-equivariant structure by letting T act on

$E^{\underline{d}}[2\dim {}^n\mathcal{F}_{\underline{d}}]$ via the character.

For a partition $\lambda \in \mathcal{P}_n^d$, let $\mathcal{O}_{\lambda^t} \subseteq {}^n\mathcal{N} \subseteq \mathfrak{gl}_d$ be the nilpotent

matrices with Jordan blocks given by λ^t.

Thm $\exists$ canonical T-equiv. isomorphism

$$(\mu_n)_* {}^n\mathcal{L}_E \simeq \bigoplus_{\substack{\text{partitions} \\ \lambda \in \mathcal{P}_n^d}} IC_{\overline{\mathcal{O}_{\lambda^t}}} \otimes \underset{\substack{\nearrow \\ \text{highest weight } \lambda \text{ irrep of } GL_n(\mathbb{C})}}{V_\lambda}$$

Remark: 1) decompose according to the torus T-weight,

we get.

$$(\mu_n)_*\left({}^n\mathcal{L}_{\underline{E}}|_{{}^n\mathcal{U}_{\underline{d}}}\right) \simeq \bigoplus_{\lambda} IC_{\overline{\mathcal{O}_{\lambda^t}}} \otimes (V_\lambda)_{\underline{d}}$$

weight space.

2). let $x \in \mathcal{O}_{\lambda^t}$, then

$$H(\mu_n^{-1}(x)) \simeq V_\lambda, \text{ and}$$

$$H(\mu_n^{-1}(x) \cap {}^n\mathcal{F}_{\underline{d}}) \simeq (V_\lambda)_{\underline{d}}.$$

this recovers the previous results

Now let's prove the theorem.

$\widetilde{gl}_d \xrightarrow{\mu} gl_d$, Springer sheaf $Spr := \mu_* \mathbb{C}_{\widetilde{gl}_d}[\dim \widetilde{gl}_d]$

We already proved

$$\mathrm{End}(Spr) = \mathbb{C}[S_d].$$

For any rep ρ of S_d, let

$$\mathcal{S}_\rho := (\rho \otimes \mathcal{S}_{pr})^{S_d}.$$

Let $\mathbb{F} :=$ Fourier transform on $\mathrm{Perv}_{mon}(\mathcal{Y})$.

Recall we have proved:

Suppose $\rho = W_\lambda \in \mathrm{Irrep}(S_d)$ for some partition λ of d, then $\mathbb{F}(\mathcal{S}_\rho) = IC_{\overline{\mathcal{O}}_{\lambda^t}}$.

Hence,

$$\bigoplus_{\lambda \in P_n^d} IC_{\mathcal{O}_{\lambda^t}} \otimes V_\lambda$$

$$\simeq \bigoplus_{\lambda \in P_n^d} \mathbb{F}(\mathcal{S}_\rho) \otimes V_\lambda$$

$$\simeq \mathbb{F}(\mathcal{S}_{E^{\otimes d}}).$$

Thus, we only need to show

$$(\mu_n)_* {}^n\mathcal{L}_E \simeq \mathbb{F}(\mathcal{S}_{E^{\otimes d}}) \qquad (*).$$

Let ${}^n\tilde{\mathfrak{g}} := \{(F^\cdot, x) \in {}^n\mathcal{F} \times \mathfrak{gl}_d \mid x(F_i) \subseteq F_i\}$

$$\begin{tikzcd} & {}^n\tilde{\mathfrak{g}} \arrow[dl, "p"'] \arrow[d, "q"] \\ \mathfrak{gl}_d. & {}^n\mathcal{F}. \end{tikzcd}$$

As in the complete flag variety case, p is a small map.

define a perverse sheaf K_E on ${}^n\tilde{\mathfrak{g}}$ as follows:

$$\forall \underline{d} \in \mathcal{P}_n^d, \quad K_E^{\underline{d}}\big|_{{}^n\tilde{\mathfrak{g}}_{\underline{d}}} := E^{\underline{d}}[\dim \mathfrak{gl}_d] \quad \text{constant sheaf}$$

Lemma: $\mathbb{F}(\mu_{n*}({}^n\mathcal{L}_E)) = p_* K_E.$

Pf:

$$\begin{tikzcd} & {}^n\mathcal{F} \times \mathfrak{gl}_d & \\ {}^n\tilde{\mathcal{N}} \arrow[ur, hook] \arrow[dr] & & {}^n\tilde{\mathfrak{g}} \arrow[ul, hook'] \arrow[dl] \\ & {}^n\mathcal{F} & \end{tikzcd}, \qquad \begin{array}{l} {}^n\tilde{\mathcal{N}} \simeq ({}^n\tilde{\mathfrak{g}})^{\perp} \\ (\mathfrak{gl}_d)^* \simeq \mathfrak{gl}_d \quad \text{trace pairing} \end{array}$$

$\Rightarrow \pi_*({}^n\mathcal{L}_E) \simeq K_E.$

Then just apply $(pr_{\mathfrak{gl}_d})_*$ to both sides, where

$$pr_{\mathfrak{gl}_d} : {}^n\mathcal{F} \times \mathfrak{gl}_d \to \mathfrak{gl}_d. \qquad \square$$

Thus, $(*)$ reduces to show

$$p_* K_E \simeq \mathcal{S}_{E^{\otimes d}}. \quad \cdots\cdots (**)$$

Since p is small, $p_* K_E \simeq IC(\mathfrak{gl}_d, p_* K_E|_{\mathfrak{gl}_d^{rs}})$.

Also, $\mathcal{S}_{E^{\otimes d}} \simeq IC(\mathfrak{gl}_d, \mathcal{S}_{E^{\otimes d}}|_{\mathfrak{gl}_d^{rs}})$.

Thus, we only need to construct a T-equiv. isomorphism of $\mathcal{S}_{E^{\otimes d}}$ and $p_* K_E$ over $\mathfrak{gl}_d^{rs}$.

Consider
$$\begin{tikzcd} {}^n\underline{\tilde{\mathfrak{g}}_d^{rs}} \arrow[r, "e_d"] \arrow[d, "p"'] \arrow[dr, phantom, "\square"] & {}^\circ\mathbb{C}^{\underline{(d)}} \arrow[d, "\underline{G_d}"] \\ \mathfrak{gl}_d^{rs} \arrow[r, "e_d"] & {}^\circ\mathbb{C}^{(d)} \end{tikzcd}$$

where ${}^{\circ}\mathbb{C}^d = \{ (z_i) \in \mathbb{C}^d \mid z_i \neq z_j \}$

$${}^{\circ}\mathbb{C}^{(d)} = {}^{\circ}\mathbb{C}^d / S_d.$$

$${}^{\circ}\mathbb{C}^{(\underline{d})} = {}^{\circ}\mathbb{C}^d / S_{\underline{d}} \quad {}^{\shortparallel}\, S_{d_1} \times S_{d_2} \times \cdots \times S_{d_n}$$

e_d = take eigenvalues

$e_{\underline{d}}(F, x)$ = eigenvalues of x on $F_1, F_2/F_1, \cdots, F_n/F_{n-1}$.

Let S_E = d-th symm. power of the local system $\mathcal{L}$ with fiber E on $\mathbb{C}$.

$$\mathrm{Sym}^d : \mathbb{C}^d \longrightarrow \mathbb{C}^{(d)} := \mathbb{C}^d / S_d$$

$$S_E := \left(\mathrm{Sym}^d_*(\mathcal{L}^{\boxtimes d}) \right)^{S_d}.$$

S_E has fiber $E^{\otimes d}$ over ${}^{\circ}\mathbb{C}^{(d)}$.

thus $e_d^* S_E[d^2] = S_{E^{\otimes d}}$.

Let $K_E^{\underline{d}} |_{{}^{\circ}\mathbb{C}^{(\underline{d})}} = E^{\underline{d}} |_{{}^{\circ}\mathbb{C}^{(\underline{d})}}$. then

$$e_{\underline{d}}^* K_E^{\underline{d}}[d^2] \simeq K_E |_{\tilde{\mathfrak{g}}_{\underline{d}}^{rs}}.$$

Then $(\star\star)$ over $\mathfrak{gl}_d^{rs}$. follows from the following lemma.

Lemma: $\exists$ T-equiv. isomorphism

$$\bigoplus_{\underline{d} \in P_n^d} (\sigma_{\underline{d}})^* K_E^{\underline{d}} \simeq S_E .$$

pf: follows from the linear alg fact

$$\bigoplus_{\underline{d} \in P_n^d} \left| S_d / S_{\underline{d}} \right| \cdot E^{\underline{d}} \simeq E^{\otimes d} . \qquad \square$$

This finishes the proof of the main theorem.

§. Lagrangian construction of the $(\mathfrak{gl}_n, \mathfrak{gl}_m)$-duality following Weiqiang Wang.

$\mathfrak{gl}_n, \mathfrak{gl}_m \curvearrowright \mathbb{C}^n \otimes \mathbb{C}^m$.

Have $\Rightarrow$ $S^d(\mathbb{C}^n \otimes \mathbb{C}^m) \simeq \bigoplus_{\lambda \in P^d_{\min(m,n)}} {}^nV_\lambda \otimes {}^mV_\lambda$ as left mods.

where $P^d_k = \{$partitions of d with at most k parts$\}$

Let ${}^n\mathcal{F} := \{0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \mathbb{C}^d\}$.

$\mathcal{N}_n := \{x \in \mathrm{End}\,\mathbb{C}^d \mid x^n = 0\}$.

${}^n\mathcal{M} := \{(x, F^\cdot) \in \mathcal{N}_n \times {}^n\mathcal{F} \mid x(F_i) \subseteq F_{i-1},\ 1 \le i \le n\} \simeq T^*({}^n\mathcal{F})$.

$$\begin{array}{ccc} & {}^n\mathcal{M} & \\ \swarrow^{\mu} & & \searrow^{\pi} \\ \mathcal{N}_n & & {}^n\mathcal{F} \end{array}$$

$x \in \mathcal{N}_n$. ${}^n\mathcal{F}_x := \mu^{-1}(x)$.

$\forall n, m > 0,\ k \geq 0$.

Let ${}^nZ^m_k := \left\{ (x, F^\cdot, F'^\cdot) \in \mathcal{N}_k \times {}^n\mathcal{F} \times {}^m\mathcal{F} \;\middle|\; \begin{array}{l} x(F_i) \subseteq F_{i-1},\ 1 \le i \le n \\ x(F'_j) \subseteq F'_{j-1},\ 1 \le j \le m \end{array} \right\}$

Rmk: ${}^{n}Z^{m}_{k} \subseteq {}^{n}Z^{m}_{k+1}$, and they stablizes to

${}^{n}Z^{m} := {}^{n}\mathcal{M} \underset{\mathcal{N}}{\times} {}^{m}\mathcal{M}$ when $k \geq \min(n,m)$.

It's easy to check

$$ {}^{n}Z^{m}_{a} \circ {}^{m}Z^{k}_{b} = {}^{n}Z^{k}_{\min(a,m,b)}. $$

Hence

$$ H({}^{n}Z^{n}) \curvearrowright H({}^{n}Z^{m}_{k}) \curvearrowleft H({}^{m}Z^{m}). $$

Recall $P^{d}_{n} = \{\text{partitions of } d \text{ with at most } n \text{ parts}\}$.

$\updownarrow$ 1:1

G-orbits in $\mathcal{N}_n$, $G = GL(d, \mathbb{C})$.

$\lambda \in P^{d}_{n}$, $\lambda^{t} = (a_1, a_2, \cdots)$ transpose of λ.

$\rightsquigarrow x_\lambda$ in $\mathcal{N}_n$ consisting of Jordan blocks of size $a_1, a_2, \cdots$.

It's easy to see

$$ {}^{n}Z^{n} \circ {}^{n}\mathcal{F}_{x} = {}^{n}\mathcal{F}_{x}, \quad {}^{m}\mathcal{F}_{x} \circ {}^{m}Z^{m} = {}^{m}\mathcal{F}_{x} $$

Thm:
$$H({}^nZ^m_k) \simeq \bigoplus_{\lambda \in \mathcal{P}^d_{\min(k,n,m)}} \mathrm{Hom}(H({}^m\mathcal{F}_{x_\lambda}), H({}^n\mathcal{F}_{x_\lambda}))$$
$$= \bigoplus_{\lambda \in \mathcal{P}^d_{\min(k,m,n)}} H({}^n\mathcal{F}_{x_\lambda}) \otimes H({}^m\mathcal{F}_{x_\lambda})^\vee$$
respecting the left $H({}^nZ^n)$-action and the right $H({}^mZ^m)$-action.

Sketch: (exactly as we did for the Steinberg variety)
$$H({}^nZ^m_k) \simeq \bigoplus_{\mathcal{O}} H({}^nZ^m_{k,\le \mathcal{O}}) / H({}^nZ^m_{k,<\mathcal{O}})$$
$$\simeq \bigoplus_{\mathcal{O}} H({}^nZ^m_{k,\mathcal{O}})$$
$$\simeq \bigoplus H({}^n\mathcal{F}_{x_\lambda})_L \otimes H({}^m\mathcal{F}_{x_\lambda})_R$$
$$\simeq \bigoplus H({}^n\mathcal{F}_{x_\lambda})_L \otimes H({}^m\mathcal{F}_{x_\lambda})^\vee_L.$$
Here, $H({}^m\mathcal{F}_{x_\lambda})_L \simeq {}^mV_\lambda$ as left $\mathfrak{gl}_m$-mod.

Switching factors = Cartan involution $\Rightarrow$ $H({}^m\mathcal{F}_{x_\lambda})_R \simeq ({}^mV_\lambda)^\vee$ as right $\mathfrak{gl}_m$-mod $\square$

Recall $\mathcal{U}(\mathfrak{gl}_n) \twoheadrightarrow H({}^nZ^n)$, and

$H(F_{x_\lambda}) \simeq V_\lambda$ has highest weight λ.

Thus, we obtained.

Thm ($(\mathfrak{gl}_n, \mathfrak{gl}_m)$-duality).

$$
\begin{array}{ccccc}
\mathcal{U}(\mathfrak{gl}_n) & \circlearrowright & S^d(\mathbb{C}^n \otimes \mathbb{C}^m) & \circlearrowleft & \mathcal{U}(\mathfrak{gl}_m) \\
\big\downarrow\!\!\!\downarrow & & \| & & \big\downarrow\!\!\!\downarrow \\
H({}^nZ^n) & \circlearrowright & H({}^nZ^m) & \circlearrowleft & H({}^mZ^m) \\
\| & & \| & & \| \\
\bigoplus_{\lambda \in P_n^d} \mathrm{End}({}^nV_\lambda) & \circlearrowright & \bigoplus_{\lambda \in P^d_{\min(n,m)}} \mathrm{Hom}({}^mV_\lambda, {}^nV_\lambda) & \circlearrowleft & \bigoplus_{\lambda \in P_m^d} \mathrm{End}({}^mV_\lambda)
\end{array}
$$

Schur duality.

$\mathcal{B}$ = Complete flag variety for $GL_d(\mathbb{C})$

${}^nW := {}^n\mathcal{M} \times_{\mathcal{N}_n} \tilde{\mathcal{N}}$, $\tilde{\mathcal{N}} = T^*\mathcal{B}$, $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$

Thm (Schur duality)

$$
\begin{tikzcd}
\mathcal{U}(\mathfrak{gl}_n) \arrow[d] & \circlearrowright & (\mathbb{C}^n)^{\otimes d} \arrow[d, equal] & \circlearrowleft & \mathbb{C}[S_d] \arrow[d, equal] \\
H({}^nZ^n) \arrow[d, equal] & \circlearrowright & H({}^nW) \arrow[d, equal] & \circlearrowleft & H(Z) \arrow[d, equal] \\
\bigoplus_{\lambda \in P_n^d} \mathrm{End}({}^nV_\lambda) & \circlearrowright & \bigoplus_{\lambda \in P_n^d} {}^nV_\lambda \otimes W_\lambda & \circlearrowleft & \bigoplus_{\lambda \in P^d} \mathrm{End}(W_\lambda)
\end{tikzcd}
$$

irrep of S_d

pf: $\mathcal{B}$ is a conn. component of ${}^d\mathcal{F}$ ass. to the partition

$$\underbrace{(1, 1, \cdots, 1)}_{d}.$$

For any $\mathfrak{gl}_d$-module U, let $U^{\mathfrak{h}_d, \det}$ denote the weight space of $wt = (1, 1, \cdots, 1)$ w.r.t. to standard basis in $\mathfrak{h}_d$.

Then $\quad H({}^nW) \simeq H({}^nZ^d)^{\mathfrak{h}_d, \det}$

(follows from the construction of $\Theta(\mathfrak{h}_d)$).

On the other hand, Howe proved

$$(\mathbb{C}^n)^{\otimes d} \simeq \left(S^d(\mathbb{C}^n \otimes \mathbb{C}^d)\right)^{\mathfrak{h}_d, \det}$$

as $(\mathfrak{gl}_n, S_d)$-modules.

We already proved

$$H({}^n Z^d) \simeq S^d(\mathbb{C}^n \otimes \mathbb{C}^d).$$

Thus $H({}^n W) \simeq (\mathbb{C}^n)^{\otimes d}$.

Similarly, $H(\mathcal{B}_{x_\lambda}) \simeq H({}^d \mathcal{F}_{x_\lambda})^{\mathfrak{h}_d, \det} = ({}^d V_\lambda)^{\mathfrak{h}_d, \det}$.

S_d acts on the zero-weight space of a $\mathfrak{gl}_d$-module, moreover

$\forall \lambda \in \mathcal{P}^d_n$, $({}^d V_\lambda)^{\mathfrak{h}_d, \det} \simeq W_\lambda \in \mathrm{Rep}(S_d)$.

Thus $H({}^n W) \simeq H({}^n Z^d)^{\mathfrak{h}_d, \det}$

$$= \bigoplus_{\lambda \in \mathcal{P}^d_n} H({}^n \mathcal{F}_{x_\lambda}) \otimes H(\mathcal{B}_{x_\lambda})$$

$$= \bigoplus_{\lambda} {}^n V_\lambda \otimes W_\lambda. \qquad \square$$

Rmk: Similarly,

$$\begin{aligned} H(2) &\simeq H({}^d2^d)^{h_d \oplus h_d,\ \det \oplus \det} \\ &= \Big(\bigoplus_{\lambda \in P^d} {}^dV_\lambda \otimes {}^dV_\lambda^\vee \Big)^{h_d \oplus h_d,\ \det \oplus \det} \\ &= \bigoplus_\lambda S_\lambda \otimes S_\lambda^\vee \\ &\simeq \mathbb{C}[S_d] \end{aligned}$$