1. Equivariant sheaves

G linear alg group$/\mathbb{C}$, X a G-variety

$$G\times X \underset{p}{\overset{a}{\rightrightarrows}} X \qquad a = \text{action}, \quad p = \text{projection}.$$

a function $f \in \mathbb{C}[X]$ is G-inv if

$$f(gx) = f(x), \quad \forall g \in G, \ x \in X. \qquad (*)$$

Hence, $f(g_1(g_2x)) = f(x) = f((g_1g_2)\cdot x)$, $g_1, g_2 \in G$. $(**)$

$(*) \Leftrightarrow a^*f = p^*f.$

$(**) \Leftrightarrow$?

$$G\times G\times X \underset{id_G\times a}{\overset{m\times id_X}{\rightrightarrows}} G\times X \underset{a}{\overset{p}{\rightrightarrows}} X$$

$(**) \Leftrightarrow (m\times id_X)^*\circ p^* \simeq (id_G\times a)^*\circ a^*.$

How about for sheaves, instead of functions?

$u: Y \to X$, $\mathcal{F} \in Sh(X)$, $u^{-1}\mathcal{F}$ = sheaf-theoretic pullback.

If $\mathcal{F} \in Sh(\mathcal{O}_X\text{-mod})$, $u^*\mathcal{F} := \mathcal{O}_Y \otimes_{u^{-1}\mathcal{O}_X} u^{-1}\mathcal{F} \in Sh(\mathcal{O}_Y\text{-mod})$.

<u>Def</u>: A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X is called G-equivariant if a) $\exists$ a given isomorphism

$$I: a^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F}.$$

b) $p_{23}^* I \circ (id_G \times a)^* I = (m \times id_X)^* I$.

$p_{23}: G \times G \times X \to G \times X$.

<u>Rmk</u>: b) means

$$\begin{array}{ccc}
(a \circ (id_G \times a))^* \mathcal{F} & \xrightarrow{(id_G \times a)^* I} & (p \circ (id_G \times a))^* \mathcal{F} = (a \circ p_{23})^* \mathcal{F} \\
\| & & \downarrow p_{23}^* I \\
(a \circ (m \times id_X))^* \mathcal{F} & \xrightarrow{(m \times id_X)^* I} & (p \circ (m \times id_X))^* \mathcal{F} = (p \circ p_{23})^* \mathcal{F}
\end{array}$$

<u>Example</u>: $\mathcal{O}_X$ has a canonical G-equiv. structure.

$$p^*\mathcal{O}_X \simeq \mathcal{O}_{G \times X} \simeq a^*\mathcal{O}_X.$$

Observation: If $\mathcal{F}$ is a locally free sheaf, i.e. a vector bundle on X. Giving a G-equiv. structure on $\mathcal{F}$ is the same as giving a G-action $\Phi: G \times \mathcal{F} \to \mathcal{F}$, s.t.

a) $\pi: \mathcal{F} \to X$ commutes with the G-actions. In particular, g takes $\mathcal{F}_x$ to $\mathcal{F}_{g.x}$.

b) $\Phi(g, \cdot): \mathcal{F}_x \to \mathcal{F}_{g.x}$ is a linear map of vector spaces.

<u>Non-example</u>:

$\mathcal{O}(1) \in \mathrm{Coh}(\mathbb{P}^1)$ does Not have a non-trivial $PGL(2,\mathbb{C})$-equiv. structure, as $PGL(2,\mathbb{C})$ have no irr. 2-dim'l rep.

<u>Thm</u>: X smooth (or more generally, normal) G-variety. $L \in \mathrm{Pic}(X)$. Then $L^{\otimes n}$ admits a G-equiv. structure for some $n \in \mathbb{Z}_{>0}$. (possibly not unique)

<u>Ex</u> $\mathcal{O}_{\mathbb{P}^1}(1)^{\otimes 2}$ has a $PGL(2,\mathbb{C})$-equiv. structure.

Prop: X smooth (more generally, normal) quasi-projective G-variety. Then any G-equiv. coherent sheaf $\mathcal{F}$ on X is a quotient of a G-equiv. locally free sheaf.

Sketch: find a projective variety $\overline{X} \supseteq X$, $\mathcal{L} \in \mathrm{Pic}(\overline{X})$ ample.

assume n is large enough, s.t.

$\mathcal{L}^{\otimes n}|_X$ has a G-equiv. structure. and

$\mathcal{F} \otimes (\mathcal{L}^{\otimes n}|_X)$ is generated by a finite number of global sections, contained in a f.d. v.s. $V \subseteq \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}|_X)$. Then

$$V \otimes (\mathcal{L}^*|_X)^{\otimes n} \twoheadrightarrow \mathcal{F}.$$

□

Cor: X smooth, quasi-projective G-variety. Then any G-equiv. coherent sheaf $\mathcal{F}$ on X has a finite locally free G-equiv. resolution.

pf: By the above prop, $\exists\ \mathcal{F}' \hookrightarrow \mathcal{F}_1 \twoheadrightarrow \mathcal{F}$, $\mathcal{F}_1$ G-equiv, locally free, $\mathcal{F}' = \mathrm{Ker}(\mathcal{F}_1 \twoheadrightarrow \mathcal{F})$. Continue this way, we get

$$\cdots \to \mathcal{F}^{n+1} \to \mathcal{F}^n \to \cdots \to \mathcal{F}^1 \twoheadrightarrow \mathcal{F} \to 0.$$

$\mathcal{F}^i$ G-equiv. locally free. We need to show the sequence can be made finite. This follows from Hilbert's Syzygy theorem.

□

<u>Thm</u> (Hilbert's Syzygy theorem)

X smooth, n-dim'l. $\mathcal{F} \in Coh(X)$, Suppose we are given a locally free resolution

$$\cdots \to \mathcal{F}^{n+1} \to \mathcal{F}^n \to \cdots \to \mathcal{F}^1 \twoheadrightarrow \mathcal{F} \to 0.$$

Then $\mathrm{Ker}(\mathcal{F}^n \to \mathcal{F}^{n-1})$ is locally free.

2. Basic Constructions in equiv. K-theory.

Let $Coh^G(X)$ = Category of G-equiv. sheaves on X.

$$K^G(X) := K_0(Coh^G(X)).$$
$$= \text{Grothendieck group of } Coh^G(X)$$

Rmk: Quillen also $\exists\ K_i^G(X) := K_i(Coh^G(X))$.

a) X = pt

$$Coh^G(pt) = Rep^{fd}(G).$$
$$K^G(pt) = R(G) := K_0(Rep^{fd}(G))$$

If G is reductive,

$$\mathbb{C} \otimes_{\mathbb{Z}} R(G) \xrightarrow{\sim} \mathcal{O}(G)^G$$

R(G) has a basis formed by the simple G-modules.

· If G is unipotent, then Lie-Engel theorem tells us that R(G) is 1-dim'l, generated by the trivial rep.

But $\mathcal{O}(G)^G$ is much larger. Thus, $\mathbb{C} \otimes_{\mathbb{Z}} R(G) \not\cong \mathcal{O}(G)^G$.

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b) Pullback

$f: Y \to X$ G-equiv.

i) if f is an open embedding or more generally if f is flat.

then $f^*: Coh^G(X) \to Coh^G(Y)$

$$\mathcal{F} \mapsto f^*\mathcal{F} := \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{F}.$$

is exact.

Hence, $f^*: K^G(X) \to K^G(Y)$.

ii). $f: Y \hookrightarrow X$ closed embedding, X, Y smooth.

$[\mathcal{F}] \in K^G(X)$,

define $f^*([\mathcal{F}]) := \sum_i (-1)^i [Tor_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{F})]$.

(we restrict to the smooth case, so that $Tor_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{F}) = 0$ if i is big enough).

c). Restriction with supports.

$f: Y \hookrightarrow X$ as in b).

$Z \subseteq X$ closed, G-stable, possibly singular.

define $f^*: K^G(Z) \to K^G(f^{-1}(Z)) = K^G(Y \cap Z)$ as follows

$\mathcal{E} \in Coh^G(Z)$. $\mathcal{F} := i_* \mathcal{E}$, $i: Z \hookrightarrow X$.

apply the construction in b) to $\mathcal{F}$,

$$\sum (-1)^i [Tor_i^{\mathcal{O}_X}(\mathcal{O}_Y, i_* \mathcal{E})]$$

Each term is supp. on $Supp\, \mathcal{O}_Y \cap Supp\, \mathcal{E} = Y \cap Z$, and they're $\mathcal{O}_Y$-mods. But they may not be $\mathcal{O}_{Y\cap Z}$-mods, as they may not be annihilated by $I_{Y\cap Z}$. But they're killed by $I_{Y \cap Z}^k$ for k big enough, since they're supp. on $Y \cap Z$.

Thus, for each term $A_i := Tor_i^{\mathcal{O}_X}(\mathcal{O}_Y, i_* \mathcal{E})$,

$$gr A_i := \sum_j [I_{Y\cap Z}^j A_i / I_{Y\cap Z}^{j+1} A] \in K^G(Y \cap Z)$$

and define $f^*[\mathcal{E}] := \sum_i (-1)^i \, gr\, Tor_i^{\mathcal{O}_X}(\mathcal{O}_Y, i_*\mathcal{E}) \in K^G(Z\cap Y)$

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d) tensor product.

X, Y G-varieties.

$\exists$ exact functor $Coh^G(X) \times Coh^G(Y) \xrightarrow{\boxtimes} Coh^G(X\times Y)$

$$(\mathcal{F}, \mathcal{F}') \mapsto p_X^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} p_Y^*\mathcal{F}'.$$

$\rightsquigarrow \boxtimes : K^G(X) \times K^G(Y) \longrightarrow K^G(X\times Y).$

(d1) X smooth G-variety. $\Delta : X \hookrightarrow X\times X$ diagonal

$\mathcal{F}, \mathcal{F}' \in Coh^G(X)$

$[\mathcal{F}] \otimes [\mathcal{F}'] := \Delta^*([\mathcal{F} \boxtimes \mathcal{F}']).$

$(K^G(X), \otimes)$ a commutative ass. $R(G)$-alg.

(d2) Tensor product with supports.

(analog of intersecting pairing with supports in homology)

X smooth, $Z, Z' \subseteq X$ closed, G-stable

define $\otimes : K^G(Z) \times K^G(Z') \longrightarrow K^G(Z \cap Z')$

by applying restriction with supp. to

$$\begin{array}{ccc} X & \overset{\Delta}{\hookrightarrow} & X \times X \\ \cup & & \cup \\ Z \cap Z' & \hookrightarrow & Z \times Z' \end{array}$$

③ Tensor product with a vector bundle.

X quasi-proj. G-variety, $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ a G-equiv. vector bund.

$E \otimes_{\mathcal{O}_X} - : Coh^G(X) \to Coh^G(X)$ is exact

$\Rightarrow E \otimes_{\mathcal{O}_X} - : K^G(X) \to K^G(X)$

e) pushforwards

$f: X \to Y$ proper, G-equiv. between two arbitrary quasi-proj.

G-varieties.

$$\exists \; K^G(X) \to K^G(Y)$$

$$[\mathcal{F}] \mapsto \sum (-1)^i [R^i f_* \mathcal{F}].$$

denote this map by f_*.

f). long exact sequence.

$i: X \hookrightarrow Y$ closed, $j: U := Y \setminus X \hookrightarrow Y$.

$\exists$ long exact sequence

$$\cdots \to K_i^G(X) \to K_i^G(Y) \to K_i^G(U) \to K_{i-1}^G(X) \to \cdots$$

f) equivariant descent.

$\pi : P \to X$ a principal G-bundle,

$\pi^* : K(X) \xrightarrow{\sim} K^G(P)$

g). Induction.

$H \subseteq G$ closed alg. subgroup, X a H-variety.

$H \circlearrowright G \times X$, $h(g, x) = (gh^{-1}, hx)$

$G \times_H X := G \times X / H$

$\downarrow pr_1$, fiber $= X$.

G/H.

$\exists$ exact functor $res : Coh^G(G \times_H X) \to Coh^H(X)$

$\mathcal{G} \longmapsto \mathcal{G}|_{e \times X}$.

$\exists$ inverse functor

$Ind_H^G : Coh^H(X) \to Coh^G(G \times_H X)$ defined as follows.

Let $p: G\times X \to X$ projection.

$\mathcal{F} \in Coh^H(X)$, then $p^*\mathcal{F} \in Coh^H(G\times X)$

$$\updownarrow \qquad \text{is by equivariant descent.}$$

$$Ind_H^G \mathcal{F} \in Coh(G\times_H X)$$

Moreover, by definition, $\exists$ obvious G-equiv. structure on $p^*\mathcal{F}$, which induces a G-equiv. structure on $Ind_H^G \mathcal{F}$.

Then $K_i^H(X) \underset{res.}{\overset{Ind_H^G}{\rightleftarrows}} K_i^G(G\times_H X)$ gives the isomorphism.

b) reduction.

Any alg group $G = R \ltimes U$, R reductive, U = unipotent radical.

Then $\exists$ forgetful map $K_i^G(X) \to K_i^R(X)$

In fact, this is an equivalence.

$$p \boxtimes a : G \times_R X \xrightarrow{\sim} G/R \times X.$$

$$\rightsquigarrow K^R(X) \xrightarrow[\text{Induction}]{\sim} K^G(G \times_R X) \xrightarrow{\sim} K^G(G/R \times X)$$

$$\uparrow p^* \qquad\qquad \mathcal{O}_{G/R} \boxtimes \mathcal{F} \uparrow \mathcal{F}$$

$$K^G(X)$$

$$(G/R) \times X \xrightarrow{p} X$$

We show p is an isomorphism.

$G/R \simeq U$.

If U is abelian, i.e $U \simeq \mathbb{C}^n$. p is an vector bundle.

Thom isomorphism (will be introduced later) shows

p^* is an isomorphism.

In general, let $U^i := [U^{i-1}, U^{i-1}]$, $U^0 = U$, $i \geq 1$

$$U \supseteq U^1 \supseteq U^2 \supseteq \dots \supseteq U^n = \{e\}$$

Each U^i is G-stable, and U^i / U^{i+1} is abelian.

$\pi_i : U/U^i \to U/U^{i-1}$ is affine bundle.

$\Rightarrow K^G(U/U_i \times X) \xrightarrow{\sim} K^G(U/U_{i-1} \times X)$.

Thus $K^G(G/R \times X) \xrightarrow{\sim} K^G(X)$.

ii). Convolution construction

(similar to the BM homology)

M_1, M_2, M_3 smooth quasi-proj. G-varieties.

$Z_{12} \subseteq M_1 \times M_2$, $Z_{23} \subseteq M_2 \times M_3$ G-stable closed.

$p_{13}: p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3$ proper.

$[\mathcal{F}_{12}] \in K^G(Z_{12})$, $[\mathcal{F}_{23}] \in K^G(Z_{23})$

$p_{12}^*[\mathcal{F}_{12}] \otimes p_{23}^*[\mathcal{F}_{23}] \in K^G(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}))$

(tensor product with supports)

$[\mathcal{F}_{12}] * [\mathcal{F}_{23}] := (p_{13})_* (p_{12}^*[\mathcal{F}_{12}] \otimes p_{23}^*[\mathcal{F}_{23}]) \in K^G(Z_{12} \circ Z_{23})$

Fact: for smooth X, $K^G(X_\Delta) \otimes K^G(X_\Delta) \xrightarrow{\otimes = *} K^G(X_\Delta)$, $X_\Delta \hookrightarrow X \times X$

j). Pairing.

X projective G-variety, $p: X \to pt.$

$$p_*[\mathcal{F}] = \sum (-1)^i H^i(X, \mathcal{F}) =: \chi(X, \mathcal{F}) \in R(G)$$

define a $R(G)$-bilinear pairing

$$K^G(X) \times K^G(X) \to R(G)$$

$$([\mathcal{F}], [\mathcal{F}']) \mapsto \langle [\mathcal{F}], [\mathcal{F}'] \rangle := p_*([\mathcal{F}] \otimes [\mathcal{F}'])$$

<u>Lemma</u>: M_1, M_2, M_3 smooth, $Y', Y'' \subseteq M_2$ two closed G-stable subvarieties, s.t. $Y' \cap Y''$ is compact.

$[\mathcal{F}_1] \in K^G(M_1)$, $[\mathcal{F}_3] \in K^G(M_3)$, $[\mathcal{G}'] \in K^G(Y')$, $[\mathcal{G}''] \in K^G(Y'')$

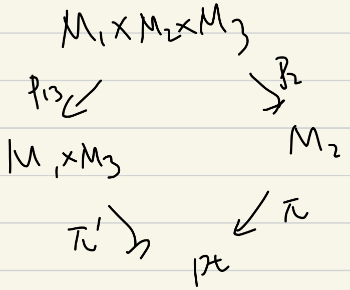
we have

$$([\mathcal{F}_1] \boxtimes [\mathcal{G}']) * ([\mathcal{G}''] \boxtimes [\mathcal{F}_3])$$

$$= \langle [\mathcal{G}'], [\mathcal{G}''] \rangle \cdot ([\mathcal{F}_1] \boxtimes [\mathcal{F}_3]).$$

<u>Rmk</u>: $\langle \mathcal{G}', \mathcal{G}'' \rangle$ is defined via $K^G(Y') \times K^G(Y'') \xrightarrow{\otimes} K^G(Y' \cap Y'') \to K^G(pt)$

pf: $p_{ij}: M_1\times M_2\times M_3 \to M_i\times M_j$, $\quad p_i: M_1\times M_2\times M_3 \to M_i$.



$$\left([\mathcal{F}_1]\boxtimes[\mathcal{G}'] \right) * \left([\mathcal{G}'']\boxtimes[\mathcal{F}_3] \right)$$

$$= (p_{13})_*\left(p_{12}^{*} \downarrow \otimes p_{23}^{*} \downarrow \right)$$

$$= (p_{13})_*\left(p_{13}^{*}([\mathcal{F}_1]\boxtimes[\mathcal{F}_3]) \otimes p_2^{*}([\mathcal{G}']\boxtimes[\mathcal{G}'']) \right)$$

$$= \left([\mathcal{F}_1]\boxtimes[\mathcal{F}_3]\right) \otimes p_{13*}\, p_2^{*}\left([\mathcal{G}']\boxtimes[\mathcal{G}'']\right)$$

$$= \langle [\mathcal{G}'], [\mathcal{G}''] \rangle \cdot \left([\mathcal{F}_1]\otimes[\mathcal{F}_3]\right).$$ □

h). Projection formula and base change.

$f: X\to Y$ proper, $\mathcal{F}\in Gh^G(X)$, $\mathcal{E}$ equivariant locally free sheaf on Y.

Then
$$f_*(\mathcal{F}\otimes f^*\mathcal{E}) = f_*(\mathcal{F})\otimes \mathcal{E}.$$

base change

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{\phi}} & Z \\ \tilde{f}\downarrow & \lrcorner & \downarrow f \\ \tilde{S} & \xrightarrow[\phi]{} & S \end{array}$$

assume:

a) Either ϕ is flat or

b) $\phi: \tilde{S} \to S$ is a closed embedding of smooth varieties, and $\exists$ a smooth fibration $f: X \to S$, s.t. $f: Z \to S$ is its restriction to a closed subset $Z \subseteq X$.

In either case, we can define $\tilde{\phi}^*$.

a) $\tilde{\phi}$ is flat

b) $\tilde{X} := \tilde{S} \times_S X \longrightarrow X$ $\qquad \tilde{\phi}^*$ = restriction with supports

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \cup & & \cup \\ \tilde{Z} & & Z \end{array}$$

Prop: If either a) or b) holds, and f is proper, then

$$\begin{array}{ccc} K^G(\tilde{Z}) & \xleftarrow{\tilde{\phi}^*} & K^G(Z) \\ \downarrow \tilde{f}_* & \circlearrowleft & \downarrow f_* \\ K^G(\tilde{S}) & \xleftarrow[\phi^*]{} & K^G(S) \end{array}$$