Example: $G = GL(V)$, $X = \mathbb{P}(V) = (V \setminus \{0\})/\mathbb{C}^*$, $n = \dim V$

$$K^G(X) = K^{G\times\mathbb{C}^*}(V\setminus\{0\}).$$

$$V\setminus\{0\} \hookrightarrow V \overset{i}{\hookleftarrow} \{0\}.$$

$$\begin{array}{ccccccc} K^{G\times\mathbb{C}^*}(\{0\}) & \xrightarrow{i_*} & K^{G\times\mathbb{C}^*}(V) & \longrightarrow & K^{G\times\mathbb{C}^*}(V\setminus\{0\}) = K^G(X) & \to & 0. \\ \| & & \wr\| & & & & \\ R(G\times\mathbb{C}^*) & \xrightarrow{i_*} & R(G\times\mathbb{C}^*) \simeq R(G)[s^{\pm 1}], & & & & \end{array}$$

s = standard rep of $\mathbb{C}^*$

Thus, $K^G(X) \simeq \operatorname{Coker} i_*$.

Thus, we only need to compute $i_* \mathcal{O}_0$

if $n=1$,

$$0 \to \mathcal{O}_V \xrightarrow{x} \mathcal{O}_V \to \mathcal{O}_0 \to 0.$$

if $n=2$

$$0 \to \mathcal{O}_V \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \mathcal{O}_V^{\oplus 2} \xrightarrow{(x,y)} \mathcal{O}_V \to \mathcal{O}_0 \to 0$$

$x, y \in V^*$, $\mathcal{O}_V^{\oplus 2} \simeq \mathcal{O}_V \otimes V^* \xrightarrow{x\otimes\frac{d}{dx} + y\otimes\frac{d}{dy}} \mathcal{O}_V$

In general, we have the Koszul complex, $d=\sum_{i=1}^{n} x_i \otimes \frac{d}{dx_i}$.

$$\underset{s^{-n}}{\mathcal{O}_V\otimes\Lambda^n V^*} \xrightarrow{d} \underset{s^{-(n-1)}}{\mathcal{O}_V\otimes\Lambda^{n-1} V^*} \xrightarrow{d} \cdots \to \underset{s^{-2}}{\mathcal{O}_V\otimes\Lambda^2 V^*} \xrightarrow{d} \underset{s^{-1}}{\mathcal{O}_V\otimes V^*} \xrightarrow{d} i_*\mathcal{O}_0 \to 0$$

This is $GL(V)$-equivariant, to add the $\mathbb{C}^*$-equivariance, notice that $\mathbb{C}^* \curvearrowright V^*$ by character s^{-1}.

Thus, $$[i_*\mathcal{O}_0] = \sum_{i=0}^{n} (-1)^i s^{-i}[\Lambda^i V^*]$$
$$= \prod_{i=1}^{n}(1-s^{-1}a_i^{-1}).$$

Where $a_1,\cdots,a_n$ are the characters of $T\subseteq GL(V)$ on V.

Thus $$K^{GL(V)}(\mathbb{P}(V)) \simeq \mathbb{Z}[s^{\pm1}][a_1^{\pm1},\cdots,a_n^{\pm1}]^{S_n} \Big/ \prod_{i=1}^{n}(1-s^{-1}a_i^{-1})$$

s = standard rep. of $\mathbb{C}^*$,

$\rightsquigarrow$

$\mathcal{O}(1)\in \mathrm{Pic}(\mathbb{P}(V))$.

§3. Koszul complex and the Thom isomorphism.

$$\begin{tikzcd} V \arrow[d, shift left, "\pi"] \\ X \arrow[u, shift left, "i"] \end{tikzcd}$$

a G-equiv. vector bundle

construct a canonical resolution of $i_*\mathcal{O}_X$.

$V^\vee =$ dual vector bundle. on X

$$\cdots \to \pi^*(\Lambda^k V^\vee) \xrightarrow{d} \pi^*(\Lambda^{k-1} V^\vee) \xrightarrow{d} \cdots \to \pi^*(\Lambda^1 V^*) \xrightarrow{d} \mathcal{O}_V \xrightarrow{\varepsilon} i_*\mathcal{O}_X \to 0 \qquad (*)$$

restriction.

The differential d is defined as follows:

d acts fiberwise. Let $v \in V$, $x = \pi(v) \in X$.

$$\Lambda^k V_x^\vee \longrightarrow \Lambda^{k-1} \Lambda_x^\vee$$

$$\check{v}_1 \wedge \cdots \wedge \check{v}_k \mapsto \sum_{j=1}^{k} (-1)^j \langle \check{v}_j, v \rangle \cdot \check{v}_1 \wedge \cdots \wedge \widehat{\check{v}_j} \wedge \cdots \wedge \check{v}_k$$

delete $\check{v}_j$.

Let $\lambda(V) := \sum (-1)^i [\Lambda^i V] \in K^G(X)$.

Prop: The complex (*) is exact. Hence,

$$i_* \mathcal{O}_X = \sum_{i=0}^{\mathrm{rk} V} (-1)^i [\pi^* \wedge^k V^\vee] = \pi^* \lambda(V^\vee).$$

pf: This is a local statement w.r.t. X. Assume $X = pt$, the complex (*) reduces to the old one.

$$0 \to \mathcal{O}_V \otimes \wedge^n V^\vee \to \mathcal{O}_V \otimes \wedge^{n-1} V^\vee \to \cdots \to \mathcal{O}_V \otimes \wedge^2 V^\vee \to \mathcal{O}_V \otimes V \to$$

$$\mathcal{O}_V \to \mathcal{O}_0 \to 0,$$

which is known to be exact. $\square$

Restriction to the zero section:

Even X is not smooth, we can still define

$$i^* : K^G(V) \to K^G(X),$$ using the finite locally free resolution of $i_* \mathcal{O}_X$ above, i.e.

$$i^*[\mathcal{F}] := \sum (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_V}(i_*\mathcal{O}_X, \mathcal{F})]$$

Thus, $i^*[\mathcal{F}]$ is computed by the following complex

$$\cdots \to \pi^*\Lambda^2 V^\vee \otimes \mathcal{F} \to \pi^*\Lambda^1 V^\vee \otimes \mathcal{F} \to \mathcal{F}.$$

Lemma: $[\mathcal{F}] \in K^G(X)$, then

$$i^*\pi^*[\mathcal{F}] = [\mathcal{F}],$$

$$i^*i_*[\mathcal{F}] = \lambda(V^\vee) \otimes [\mathcal{F}].$$

Pf: Let's prove the second one.

$i^*i_*[\mathcal{F}]$ is computed by

$$\pi^*\Lambda^k V^\vee \otimes i_*\mathcal{F}$$

$$= i_*(i^*\pi^*\Lambda^k V^\vee \otimes \mathcal{F}) \qquad \text{proj. formula}$$

$$= i_*(\Lambda^k V^\vee \otimes \mathcal{F}) \qquad \text{a sheaf supp. on } X.$$

Thus $i^* i_*[\mathcal{F}] = \sum (-1)^k [\Lambda^k V^\vee \otimes \mathcal{F}]$

$$= \lambda(V^\vee) \otimes [\mathcal{F}]. \qquad \square$$

We also have this in the non-linear setting.

<u>Prop</u>: $i: N \hookrightarrow M$ G-equiv. closed embedding of a smooth G-variety N as a submanifold of a smooth G-variety M.

Then $i^* i_* [\mathcal{F}] = \lambda(T_N^* M) \otimes [\mathcal{F}]$, $\forall\ \mathcal{F} \in K^G(N)$.

<u>Cor</u>: For a s.e.s. $V_1 \hookrightarrow V \twoheadrightarrow V_2$ in $Coh^G(X)$

$$\lambda(V) = \lambda(V_1) \otimes \lambda(V_2).$$

pf of the Cor: For a v.b. $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$, $\pi_E : E \rightleftarrows X : i_E$.

$$\begin{tikzcd} X \arrow[r, hook, "i_{V_1}"] \arrow[rr, bend right, "i_V"'] & V_1 \arrow[r, hook, "j"] & V \end{tikzcd} \qquad T^*_{V_1} V \simeq V_2^\vee.$$

$$
\begin{aligned}
i_V^*(i_V)_*\mathcal{O}_X &= i_{V_1}^* j^* j_* i_{V_1 *}\mathcal{O}_X \\
\| \qquad &= i_{V_1}^*\left(\pi_{V_1}^*\lambda(V_2^\vee)\otimes i_{V_1 *}\mathcal{O}_X\right) \\
\lambda(V^\vee) \qquad & \\
&= i_{V_1}^*\left(i_{V_1 *}\left(\mathcal{O}_X \otimes i_{V_1}^*\pi_{V_1}^*\lambda(V_2^\vee)\right)\right) \\
&= i_{V_1}^*\left(i_{V_1 *}\left(\lambda(V_2^\vee)\right)\right) \\
&= \lambda(V_2^\vee)\otimes\lambda(V_1^\vee)
\end{aligned}
$$

$\square$

For the proof of the prop, we need to use the deformation to the normal bundle diagram.

$$
\begin{tikzcd}
N \arrow[r, hook] \arrow[d, hook] & N\times\mathbb{C} \arrow[d, hook] & N\times\mathbb{C}^* \arrow[l, hook'] \arrow[r] \arrow[d] & N \arrow[d, hook] \\
T_N M \arrow[r, hook] \arrow[d] & X_N \arrow[d] & M\times\mathbb{C}^* \arrow[l, hook'] \arrow[r] \arrow[d] & M \\
\{0\} \arrow[r, hook] & \mathbb{C} & \mathbb{C}^* \arrow[l, hook']
\end{tikzcd}
$$

to reduce to the linear case.

$E \to X$ G-equiv. affine bundle on X.

<u>Thm</u> (Thom isomorphism theorem).

$\forall j \geq 0$, $\pi^* : K_j^G(X) \xrightarrow{\sim} K_j^G(E)$

<u>Cor</u> : $\pi : V \to X$ G-equiv. Vectorbundle. $i : X \hookrightarrow V$.

Then $i^* : K^G(V) \xrightarrow{\sim} K^G(X)$.

$(i^* \circ \pi^* = \mathrm{id})$.

§4. The Künneth formula, Beilinson resolution & the projective bundle theorem.

Künneth formula

X smooth proj. G-variety.

$\mathcal{O}_\Delta :=$ structure sheaf of the diagonal in $X \times X$.

For arbitrary G-variety Y, $\exists$ convolution

$$K^G(Y \times X) \otimes K^G(X) \to K^G(Y)$$

(since X smooth, any element in $Coh^G(X)$ has a finite locally free resolution, thus, $\otimes$ can be defined)

<u>Thm</u>. The followings are equivalent:

(a) The natural map $\pi : K^G(X) \otimes_{R(G)} K^G(Y) \to K^G(X \times Y)$

$$(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \boxtimes \mathcal{G}$$

is an isomorphism for arbitrary G-variety Y.

(b) $\mathcal{O}_\Delta \in K^G(X\times X)$ belongs to the image of π for $Y=X$

(c) $K^G(X)$ is a finitely generated projective $R(G)$-module, and for any G-variety Y,

$$K^G(Y\times X) \xrightarrow{\sim} \mathrm{Hom}_{R(G)}(K^G(X), K^G(Y))$$

(induced by convolution)

(d) $K^G(X)$ is a finitely generated projective $R(G)$-module,

$K^G(X\times X)$ is a finitely generated proj. $R(G)$-mod s.t.

$\mathrm{rk}\, K^G(X\times X) = (\mathrm{rk}\, K^G(X))^2$, and

$\langle -,- \rangle : K^G(X)\times K^G(X) \to R(G)$ is non-degenerate, i.e.

the induced map $K^G(X) \to (K^G(X))^\vee := \mathrm{Hom}_{R(G)}(K^G(X), R(G))$

$$F \mapsto \langle F, - \rangle$$

is an isomorphism.

Beilinson resolution

$V/\mathbb{C}$ vector space of dim $n+1$, $\mathbb{P}=\mathbb{P}(V)$.

$\varepsilon : \mathbb{P}_\Delta \hookrightarrow \mathbb{P}\times\mathbb{P}$ diagonal.

Construction: For any $v\in V$, let $\bar{v} := \mathbb{C}v \in \mathbb{P}$.

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}}(-1) & \supseteq & \mathbb{C}\cdot v \\ \downarrow & & \downarrow \\ \mathbb{P} & \ni & \bar{v} \end{array} \qquad H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \simeq V^*.$$

Recall the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-1) \to V\otimes\mathcal{O}_{\mathbb{P}} \to Q \to 0$$

$$Q = T\otimes\mathcal{O}_{\mathbb{P}}(-1), \quad T = \text{tangent sheaf}$$

$\rightsquigarrow H^0(\mathbb{P}, Q) = H^0(\mathbb{P}, V\otimes\mathcal{O}_{\mathbb{P}}) = V$

$H^0(\mathbb{P}\times\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)\boxtimes Q) \simeq V^*\otimes V \simeq \mathrm{Hom}(V, V).$

Let $s \in LHS$ be the global section of $\mathcal{O}_{\mathbb{P}}(1) \boxtimes Q$

corresponding to $id \in Hom(V, V)$

More explicity, s corresponds to a sheaf morphism

$\hat{s}: pr_1^* \mathcal{O}_{\mathbb{P}}(-1) \to pr_2^* Q$ $\qquad pr_i : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$.

$\mathcal{O}_{\mathbb{P}}(-1)|_{\bar{v}} = \mathbb{C}v$, $Q|_{\bar{w}} = V/\mathbb{C}w$.

$\hat{s}(\bar{v}, \bar{w}) : c \cdot v \mapsto c \cdot v \pmod{\mathbb{C} \cdot w}$.

Thus, $\hat{s}(\bar{v}, \bar{w}) \equiv 0$ iff $\bar{v} = \bar{w}$.

$\Rightarrow$ The zero locus of s, $Z(s) = \mathbb{P}_\Delta$.

Contractions with $s \in H^0(\mathcal{O}_{\mathbb{P}}(1) \boxtimes Q)$ gives

$$0 \to \Lambda^n (\mathcal{O}_{\mathbb{P}}(-1) \boxtimes Q^*) \to \Lambda^{n-1} (\mathcal{O}_{\mathbb{P}}(-1) \boxtimes Q^*) \to \cdots \to \mathcal{O}_{\mathbb{P}}(-1) \boxtimes Q^* \to \mathcal{O}_{\mathbb{P} \times \mathbb{P}} \twoheadrightarrow \mathcal{O}_{\mathbb{P}_\Delta}.$$

this is a locally free resolution of $\mathcal{O}_\Delta$, called the Koszul complex.

Recall $Q = T \otimes \mathcal{O}_{\mathbb{P}}(-1) \Rightarrow Q^* = T^* \otimes \mathcal{O}_{\mathbb{P}}(1) = \Omega^1_{\mathbb{P}}(1)$.

Thus, the above resolution becomes. (Beilinson resolution)

$$\boxed{\begin{aligned} 0 \to \mathcal{O}_{\mathbb{P}}(-n) \boxtimes \Omega^n_{\mathbb{P}}(n) \to \mathcal{O}_{\mathbb{P}}(-n+1) \boxtimes \Omega^{n-1}_{\mathbb{P}}(n-1) \to \cdots \to \mathcal{O}_{\mathbb{P}}(-1) \boxtimes \Omega^1_{\mathbb{P}}(1) \\ \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \to \mathcal{O}_{\mathbb{P}_\Delta} \to 0. \end{aligned}}$$

where $\Omega^k_{\mathbb{P}}(k) := (\Lambda^k \Omega^1_{\mathbb{P}}) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(k)$.

Cor: The Künneth theorem holds for $X = \mathbb{P}^n$.

Pf: $\mathcal{O}_{\mathbb{P}_\Delta}$ is in the image of $K^G(X) \otimes K^G(X) \to K^G(X \times X)$ $\square$
