The projective bundle theorem.

$$\begin{array}{c} E \\ \downarrow \\ X \end{array} \quad G\text{-equiv. rk } n \text{ v.b. on } X.$$

$\pi : \mathbb{P}(E) \to X$ projective bundle with fiber $\mathbb{P}^{n-1}$.

$$\begin{array}{c} \mathcal{O}(k) \\ \downarrow \\ \mathbb{P}(E) \end{array}$$ germs of sections = germs of regular functions on $E \setminus (\text{zero section})$ that are homogeneous of deg k along the fibers.

Thm (Quillen) $K^G(\mathbb{P}(E))$ is freely generated over $K^G(X)$ by the classes $[\mathcal{O}(k)]$, $0 \le k \le n-1$.

Pf:
$$\begin{tikzcd}
\mathbb{P} \times_X \mathbb{P} \arrow[rr, "pr_2"] \arrow[dd, "pr_1"'] & & \mathbb{P} \arrow[dd, "\pi"] \\
& \mathbb{P}_\Delta \arrow[ul, bend right, "\Sigma"'] \arrow[ur, equal, "p_2"] \arrow[dl, equal, "p_1"] \arrow[dr, "\pi_\Delta"] & \\
\mathbb{P} \arrow[rr, "\pi"'] & & X
\end{tikzcd}$$

$\exists$ relative version of the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}/X}(-1) \to \pi^* V \to Q_{\mathbb{P}/X} \to 0.$$

Same argument as above gives

$$0 \to \mathcal{O}_{\mathbb{P}/X}(-n+1) \otimes \Omega^{n-1}_{\mathbb{P}/X}(n-1) \to \cdots \longrightarrow \mathcal{O}_{\mathbb{P} \times_X \mathbb{P}} \longrightarrow \varepsilon_* \mathcal{O}_{\mathbb{P}_\Delta} \to 0.$$

If $\mathcal{K}$ is a G-equiv. sheaf on $\mathbb{P} \times_X \mathbb{P}$, define a relative version of the convolution action

$$\mathcal{K} * : K^G(\mathbb{P}) \to K^G(\mathbb{P})$$
$$\mathcal{F} \mapsto \sum (-1)^i (pr_1)_* \operatorname{Tor}_i(\mathcal{K}, pr_2^* \mathcal{F})$$

Take $\mathcal{K} = \varepsilon_* \mathcal{O}_{\mathbb{P}_\Delta}$

$$\varepsilon_* \mathcal{O}_{\mathbb{P}_\Delta} \otimes pr_2^* \mathcal{F} = \varepsilon_* (\mathcal{O}_{\mathbb{P}_\Delta} \otimes \varepsilon^* pr_2^* \mathcal{F})$$
$$= \varepsilon_* (p_2^* \mathcal{F})$$

$$\Rightarrow \varepsilon_* \mathcal{O}_{\mathbb{P}_\Delta} * \mathcal{F} = (pr_1)_* \varepsilon_* p_2^* \mathcal{F} = p_{1*} p_2^* \mathcal{F} = \mathcal{F}.$$

On the other hand, using the relative resolution of $\varepsilon_* \mathcal{O}_{\mathbb{P}_\Delta}$,

we get

$$\varepsilon_* \mathcal{O}_{\mathbb{P}_\Delta} * \mathcal{F} = \left(\mathcal{O}_{\mathbb{P} \times_X \mathbb{P}} - \mathcal{O}_{\mathbb{P}/X}(-1) \boxtimes \Omega^1_{\mathbb{P}/X}(1) + \cdots \right) * \mathcal{F}.$$

$$\left(\mathcal{O}_{\mathbb{P}/X}(-i) \boxtimes \Omega^i_{\mathbb{P}/X}(i) \right) * \mathcal{F}$$

$$= pr_{1*}\left(pr_1^* \mathcal{O}_{\mathbb{P}/X}(-1) \otimes pr_2^* \left(\Omega^i_{\mathbb{P}/X}(i) \otimes \mathcal{F} \right) \right)$$

$$= \mathcal{O}_{\mathbb{P}/X}(-i) \otimes \pi^* \pi_* \left(\Omega^i_{\mathbb{P}/X}(i) \otimes \mathcal{F} \right)$$

Thus,

$$\mathcal{F} = \left(\varepsilon_* \mathcal{O}_{\mathbb{P}_\Delta} \right) * \mathcal{F}$$

$$= \sum_{i=0}^{n-1} (-1)^i \mathcal{O}_{\mathbb{P}/X}(-i) \otimes \pi^* \pi_* \left(\Omega^i_{\mathbb{P}/X}(i) \otimes \mathcal{F} \right)$$

Let $\mathcal{E} = \mathcal{F}(n-1)$.

$$\mathcal{E} = \sum_{i=0}^{n-1} (-1)^i \mathcal{O}_{\mathbb{P}/X}(n-1-i) \otimes \pi^* \pi_* \left(\Omega^i_{\mathbb{P}/X}(i-n+1) \otimes \mathcal{E} \right)$$

This finishes the proof. $\square$

§. The Chern character map

For a smooth variety X, Let $A(X) :=$ ring of cycles modulo rational equivalence.

$\exists$ a homomorphism, called Chern character map

$$ch : K(X) \to A(X)_{\mathbb{Q}}.$$

determined by:

(i) ch is a ring homomorphism

(ii) if $f : Y \to X$, then $ch \circ f^* = f^* \circ ch$.

(iii) If $\mathcal{L}$ is a line bundle on X,

$$ch([\mathcal{L}]) = \exp(c_1(\mathcal{L}))$$

See the appendix of Hartshorne or Fulton.

If X is a closed subvariety of a smooth quasi-proj. variety M.

There is also a homology Chern character map

$ch_*: K(X) \to H_*(X, \mathbb{C})$ = Borel-Moore homology of X.

It has the following properties, see Fulton's book.

<u>Prop</u>: (i) Normalization:

$$ch_*(\mathcal{O}_X) = [X] + r \in H_*(X), \quad r \in H_{<2\dim_{\mathbb{C}} X}(X)$$

(ii) Additivity: For any s.e.s. $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$,

$$ch_*(\mathcal{F}) = ch_*(\mathcal{F}') + ch_*(\mathcal{F}'').$$

(iii) Restriction to an open subset: $U \subseteq M$ Zariski open, M smooth. $X \subseteq M$ closed. $i: X \cap U \hookrightarrow X$. Then

$$\begin{tikzcd} K(X) \arrow[r, "i^*"] \arrow[d, "ch_*"] \arrow[dr, phantom, "\circlearrowright"] & K(X \cap U) \arrow[d, "ch_*"] \\ H_*(X) \arrow[r, "i^*"] & H_*(X \cap U) \end{tikzcd}$$

Thm (Riemann-Roch for singular varieties, Baum-Fulton-MacPherson)

Given
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d, hook] \\ M \arrow[r, "\bar{f}"] & N \end{tikzcd}$$
M, N smooth.

f proper.

$$\Longrightarrow \quad Td_N \cdot ch_*(f_* \mathcal{F}) = f_*(Td_M \cdot ch_*(\mathcal{F})) \qquad \forall\, \mathcal{F} \in K(X).$$

(Recall the Todd class is defined by the function $\frac{x}{1-e^{-x}}$)

§ The localization theorem.

Let A be a complex torus, $a \in A \rightsquigarrow$ multiplicative subset

$S \subseteq R(A)$ which don't vanish at a.

$R_a := S^{-1} R(A)$.

$\forall$ R-mod M, let $M_a := R_a \otimes_R M$.

$$\begin{array}{l} E \\ \downarrow \\ X \end{array} \quad A\text{-equiv. v.b. on } X, \text{ where } A \circlearrowright X \text{ trivially.}$$

$$E = \bigoplus_{\alpha \in \operatorname{Sp} E} E_\alpha, \qquad \operatorname{Sp} E = \{A\text{-weights of } E\} \subseteq \operatorname{Hom}(A, \mathbb{C}^*)$$

$$K^A(X) \simeq R(A) \otimes_{\mathbb{Z}} K(X).$$

$$\lambda(E) := \sum (-1)^i \Lambda^i E = \sum (-1)^i \Lambda^i \Big(\sum_{\alpha \in \operatorname{Sp} E} \alpha \otimes E_\alpha \Big) \in R(A) \otimes K(X)$$

<u>Prop</u>: Assume $\forall \alpha \in \operatorname{Sp} E$, $\alpha(a) \neq 1$, s.t. $X = E^a$. Then

$$K_j^A(X)_a \xrightarrow{\lambda(E)} K_j^A(X)_a \text{ is an isomorphism.}$$

Pf: Step 1: First assume E = trivial v.b.

if $\dim E = 1$, $\operatorname{Sp} E = \{\alpha\}$, $\alpha(a) \neq 0$

then $\lambda(E) = 1-\alpha \in S \subseteq R(A)$.

Thus, $\lambda(E)$ is invertible in R_a.

In general, $E = \bigoplus_\alpha V_\alpha$. $\lambda(E) = \prod_\alpha (1-\alpha)^{\dim V_\alpha} \in S$.

Step 2: induction on $\dim X$.

$\dim X = 0$ follows from Step 1.

Assume $\dim X > 0$, $\exists$ Zariski open dense $U \subseteq X$, $Y := X \setminus U$.

s.t. $E|_U$ is trivial.

$$\begin{tikzcd}
\cdots \arrow[r] & K_j^A(Y)_a \arrow[r] \arrow[d, "\lambda(E)", "\simeq"'] & K_j^A(X)_a \arrow[r] \arrow[d, "\lambda(E)"] & K_j^A(U)_a \arrow[r] \arrow[d, "\lambda(E)", "\simeq"'] & K_{j-1}^A(Y)_a \arrow[r] \arrow[d, "\lambda(E)", "\simeq"'] & \cdots \\
\cdots \arrow[r] & K_j^A(Y)_a \arrow[r] & K_j^A(X)_a \arrow[r] & K_j^A(U)_a \arrow[r] & K_{j-1}^A(Y)_a \arrow[r] & \cdots
\end{tikzcd}$$

$\simeq$ follows from step 1, $\simeq$ follows from induction.

Thus, $\lambda(E): K_j^A(X)_a \longrightarrow K_j^A(X)_a$ is also an isomorphism.

$\square$

<u>Cor</u>:
$$\begin{tikzcd} E \arrow[d, shift right] \\ X \arrow[u, shift right, "i"'] \end{tikzcd}$$
as before. Assume $a \in A$ s.t. $E^a = X$.

Then $i_*: K_j^A(X)_a \xrightarrow{\sim} K_j^A(E)_a$.

pf: Thom isomorphism $\Rightarrow$ i^* is an isomorphism.

$i^* i_* = \lambda(E^\vee) \Rightarrow i_*$ is an iso. by the above prop. $\square$

More generally, we have.

Thm (Thomason's Localization theorem)

For any A-variety X, $i: X^a \hookrightarrow X$,

$$i_*: K^A(X^a)_a \xrightarrow{\sim} K^A(X)_a \text{ is an isomorphism.}$$

<u>Rmk</u>. $i_*. K^A(X^A) \longrightarrow K^A(X)$

$\ker i_*$ and $\operatorname{coker} i_*$ are $K_T(pt)$-modules, and have some support in T

Thomason proved.

$$\operatorname{Supp}\ker i_*,\ \operatorname{Supp}\operatorname{coker} i_* \subseteq \bigcup_\mu \{t^\mu = 1\}.$$

for finitely many non-trivial characters μ.

Since $K^A(X^A) \simeq R(A) \otimes_{\mathbb{Z}} K(X^A)$ is torsion free,

$\Rightarrow \ker i_* = 0$

Thus, i_* becomes an isomorphism by inverting those $(t^\mu - 1)$.

In literatures, people usually invert all the non-zero elements in $R(A)$, i.e. let

$$K^A(X)_{loc.} = K^A(X) \otimes_{R(A)} \operatorname{Frac} R(A)$$

Fraction field of $R(A)$.

Then $i_* : K^A(X^A)_{loc} \xrightarrow{\sim} K^A(X)_{loc}.$

§ Functoriality.

M smooth variety$/\mathbb{C}$, $G/\mathbb{C}$ reductive.

<u>Lemma</u>: M^G is a smooth subvariety of M.

($\Leftarrow$ Luna slice theorem)

Let A be a complex torus, M smooth quasi-proj. A-variety

<u>Def</u>: $a \in A$ is called M-regular if $M^A = M^a$.

<u>Rmk</u>: $N := T_{M^A} M$ normal bundle.

$A \circlearrowright N$ acts on the fiber, fixing the base M^A.

Thus $N = \bigoplus_{\alpha \in \operatorname{Sp} N} N_\alpha$.

Then $a \in A$ is M-regular iff $\alpha(a) \neq 1 \ \forall \alpha \in \operatorname{Sp} N$.

Let $\lambda_A := \sum (-1)^i \cdot \Lambda^i N^\vee \in K^A(M^A) = R(A) \otimes_{\mathbb{Z}} K(M^A)$.

$i : M^A \hookrightarrow M.$

Recall $i^* i_* = \lambda_A$. and $K_i^A(M^A)_a \xrightarrow[\sim]{\lambda_A} K_i^A(M^A)_a$

for any M-regular a.

Let $ev : K^A(pt) = R(A) \to \mathbb{C}_a$ be the evaluation homomorphism.

$$f \longmapsto f(a)$$

Let λ_a be the image of λ_A under

$$K^A(M^A) = R(A) \otimes_{\mathbb{Z}} K(M^A) \xrightarrow{ev \otimes id} \mathbb{C} \otimes_{\mathbb{Z}} K(M^A) =: K_{\mathbb{C}}(M^A)$$

i.e. $\lambda_a = \bigotimes_{\alpha \in Sp N} \left(\sum_i (-\alpha(a))^i \cdot \Lambda^i N_\alpha^\vee \right).$

Since $\lambda_A : K^A(M^A)_a \xrightarrow{\sim} K^A(M^A)_a$,

$\exists\ \lambda_a^{-1} \in K_{\mathbb{C}}(M^A)$.

Define $res_a : K^A(M) \longrightarrow K_{\mathbb{C}}(M^A)$

$$\mathcal{F} \longmapsto (\lambda_a)^{-1} \otimes ev(i^* \mathcal{F}) \in K_{\mathbb{C}}(M^A).$$

Lemma: Let a be M-regular, then

$$\mathrm{res}_a : \mathbb{C}_a \otimes_{R(A)} K^A(M) \xrightarrow{\sim} K_{\mathbb{C}}(M^A).$$

pf: Thomason Localization theorem $\Rightarrow$

$$i_* : K^A(M^A)_a \xrightarrow{\sim} K^A(M)_a.$$

tensoring with $\mathbb{C}_a$,

$$i_*^{\mathbb{C}} : K^A_{\mathbb{C}}(M^A) \xrightarrow{\sim} K^A(M) \otimes_{R(A)} \mathbb{C}_a.$$

Moreover,

$$ev \circ i^* i_* = \lambda_a.$$

$$\Rightarrow (i_*^{\mathbb{C}})^{-1} = \lambda_a^{-1} \otimes ev \circ i^* = \mathrm{res}_a.$$

$\square$

Thm $f: X \to Y$ proper, A-equivariant, X, Y smooth

Assume $a \in A$ is both X and Y regular. then

$$\begin{tikzcd} K^A(X) \arrow[r, "f_*"] \arrow[d, "\mathrm{res}_a"'] \arrow[dr, phantom, "\circlearrowleft"] & K^A(Y) \arrow[d, "\mathrm{res}_a"] \\ K_{\mathbb{C}}(X^A) \arrow[r, "f_*"'] & K_{\mathbb{C}}(X^A) \end{tikzcd}$$

Pf.

$$\begin{tikzcd} K^A(X) \arrow[r, "f_*"] \arrow[d, "\mathbb{C}_a \otimes"', "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & K^A(Y) \arrow[d, "\mathbb{C}_a \otimes"] \\ \mathbb{C}_a \otimes K^A(X) \arrow[r] \arrow[d, "\mathrm{res}_a"'] & \mathbb{C}_a \otimes K^A(Y) \arrow[d, "\mathrm{res}_a"] \\ K_{\mathbb{C}}(X^A) \arrow[r, "f_*"] & K_{\mathbb{C}}(X^A) \end{tikzcd}$$

Since $\mathrm{res}_a = (i_*^{\mathbb{C}})^{-1}$, the commutativity of the bottom square is equivalent to

$$\begin{tikzcd} \mathbb{C}_a \otimes K^A(X) \arrow[r, "f_*"] & \mathbb{C}_a \otimes K^A(Y) \\ K_{\mathbb{C}}(X^A) \arrow[r, "f_*"] \arrow[u, "i_*^{\mathbb{C}}"] \arrow[ur, phantom, "\circlearrowright"] & K_{\mathbb{C}}(Y^A) \arrow[u, "i_*^{\mathbb{C}}"'] \end{tikzcd},$$

and this follows from

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X^A \arrow[r, "f"] \arrow[u, "i"] \arrow[ur, phantom, "\circlearrowright"] & Y^A \arrow[u, "i"'] \end{tikzcd}$$

$\square$