1) general case.

G complex semisimple connected Lie group, $\supseteq B$ Borel subgp.

$\mathfrak{g} = \mathrm{Lie}\, G$, W = Weyl gp.

Recall $x \in \mathfrak{g}$ is regular if $\dim Z_{\mathfrak{g}}(x) = \mathrm{rk}\, \mathfrak{g}$

semisimple (resp. nilpotent) if $\mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g}$ is semisimple (resp. nilpotent).

Bruhat decomposition $G = \bigsqcup BwB$.

flag variety $\mathcal{B} := G/B = \{$Borel subgps / subalgs$\}$

$\mathcal{B} \subseteq Gr(\dim \mathfrak{b}, \mathfrak{g})$.

closed subvariety formed by all solvable Lie subalgs

$\Rightarrow \mathcal{B}$ is a projective variety.

Borel fixed point thm $\Rightarrow$ Plücker embedding

$\mathcal{B} \hookrightarrow \mathbb{P}(V_\lambda)$, λ non-degenerate, dominant weight.

$\mathfrak{b} \longmapsto$ the unique line in V_λ fixed by $\mathfrak{b}$.

<u>Lemma</u>:

$W \longrightarrow B\backslash G/B \longrightarrow \{B\text{-orbits on } \mathcal{B}\} \longrightarrow \left\{\begin{matrix} G\text{-diagonal} \\ \text{orbits on } \mathcal{B}\times\mathcal{B} \end{matrix}\right\}$

$w \longmapsto BwB$.

$BgB \longmapsto B\text{-orbit of } g.B$

$B\text{-orbit of } \mathfrak{b} \longmapsto G.(\mathfrak{b}_0, \mathfrak{b})$

$\mathfrak{b}_0 = \operatorname{Lie} B$.

These maps are bijections.

pf: follows immediately from the Bruhat decomposition $\square$

Chevalley restriction thm: $\mathfrak{h} \subseteq \mathfrak{g}$ any Cartan subalg.

restriction map gives an isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W.$$

Grothendieck's simultaneous resolution

$$\tilde{\mathfrak{g}} := \{(x,\mathfrak{b}) \in \mathfrak{g}\times\mathcal{B} \mid x\in\mathfrak{b}\} \simeq G\times_B \mathfrak{b}$$

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[d, "\mu"] \arrow[dr, "\pi"] & \\ \mathfrak{g}. & \mathcal{B}. \end{tikzcd}$$

$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[r, hook] \arrow[d] & \tilde{\mathfrak{g}} \arrow[d, "\mu"'] \arrow[dr, "\nu"] & \\ \mathcal{N} \arrow[r, hook] \arrow[d] & \mathfrak{g} \arrow[d, "\rho"'] & \mathfrak{h} \arrow[dl] \\ 0 \arrow[r, hook] & \mathfrak{h}/\!\!/W := \operatorname{Spec}\mathbb{C}[\mathfrak{h}]^W & \end{tikzcd}$$

Kostant.

Prop 3.2.5.

ρ is induced by $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$

$$\nu((x,\mathfrak{b})) := x \bmod [\mathfrak{b},\mathfrak{b}] \in \mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \simeq \mathfrak{h}.$$

Thm: 1). The diagrams commute.

2) $\forall\, x \in \mathfrak{h}$, $\mu : \nu^{-1}(x) \to \rho^{-1}(x)$ is a resolution of singularities

3). $\forall\, x \in \mathfrak{g}^{rs}$, $\exists$ canonical free W-action on $\mu^{-1}(x)$ making the projection $\tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ a principle W-bundle.

4) $\exists$ finitely many G-orbits on $\mathcal{N}$.

5) $\tilde{\mathcal{N}} \simeq T^*\mathcal{B} \simeq G \times_B \mathfrak{n}$, and

$\tilde{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularities.

(Springer resolution)

2). Moment maps

M a C^∞-manifold in the $\mathbb{R}$-case, or smooth alg. variety in the $\mathbb{C}$-case.

A symplectic structure on M is a non-degenerate regular 2-form ω such that $d\omega = 0$.

<u>Ex</u>: $M = \mathbb{C}^{2n}$ $\quad (q_1, \cdots, q_n, p_1, \cdots, p_n)$

$$\omega = \sum_i dp_i \wedge dq_i$$

<u>Ex</u>. $M = T^*N$, N smooth.

Construct a one-form λ on M, and set $\omega = d\lambda$.

$x \in N$, $\alpha \in T_x^*N$, $\qquad \pi : T^*N \to N$

$\pi_* : T_\alpha(T^*N) \to T_xN$, $\quad \xi \in T_\alpha(T^*N)$

$\lambda(\xi) := \langle \alpha, \pi_* \xi \rangle \in \mathbb{C}$.

$\underline{\text{Ex}}$. $G \circlearrowright \mathfrak{g}^*$ = the dual of $\mathfrak{g}$.

$\mathcal{O}$ = a coadjoint orbit $\subseteq \mathfrak{g}^*$.

claim: $\mathcal{O}$ has a natural symplectic structure.

$\alpha \in \mathcal{O}$, $\mathcal{O} \simeq G/G_\alpha$, G_α = Stabilizer of α in G

$T_\alpha \mathcal{O} \simeq \mathfrak{g}/\mathfrak{g}_\alpha$.

$\omega_\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $(x, y) \mapsto \alpha([x, y])$.

check 1) $\omega_\alpha([\mathfrak{g}, \mathfrak{g}_\alpha]) = 0$.

2) $\alpha \mapsto \omega_\alpha$ gives a 2-form ω, show $d\omega = 0$

(M, ω) a symp. manifold.

$$\begin{array}{ccc} \mathcal{O}(M) & \xrightarrow{\partial} & \text{vector fields on } M \\ \Psi & & \Psi \\ f & \longmapsto & \xi_f \end{array}$$

$\omega(\cdot, \xi_f) = df$

define a bracket $\{\cdot,\cdot\}$ on $\mathcal{O}(M)$.

$$\{f,g\} := \omega(\xi_f, \xi_g) = -\xi_g(f)$$

<u>Prop</u>: $\mathcal{O}(M)$ is a Lie alg w.r.t. $\{\cdot,\cdot\}$, and it satisfies the Leibniz rule:

$$\{f, g\cdot h\} = \{f,g\}\cdot h + g\cdot\{f,h\}.$$

(Poisson alg).

Suppose a Lie Group $G \curvearrowright M$, preserving ω.

$\rightsquigarrow$ $\mathfrak{g} \to$ symplectic vector fields on M.
(ξ, $L_\xi \omega = 0$).

We say the action is Hamiltonian if

$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, dashed, "\exists H"'] \arrow[dr] \arrow[d, phantom, "\circlearrowright"] & \\ \mathcal{O}(M) \arrow[rr, "\partial"] & & \text{Symp. vector fields on } M \end{tikzcd}$$

Define the moment map $\mu : M \to \mathfrak{g}^*$

$m \in M$, $x \in \mathfrak{g}$, $\mu(m)(x) = H_x(m)$.

$M = T^*N$, $G \circlearrowright N$.

$\mathfrak{g} \to$ Vector fields on N $\to$ Vector fields on M

$x \mapsto u_x \longmapsto \tilde{u}_x$

Claim: G action on T^*N is Hamiltonian,

with $H(x) = \lambda(\tilde{u}_x) \in \mathcal{O}(M)$,

where $\lambda =$ canonical 1-form on M above.

G Lie group, $P \leq G$ subgp.

$G \circlearrowright G/P$. $\quad T^*(G/P) \simeq G \times_P (\mathrm{Lie}\,P)^\perp \subseteq G \times_P \mathfrak{g}^*$

<u>Prop</u>: the moment map is

$$\mu : T^*(G/P) \simeq G\times_P (\mathrm{Lie}\,P)^{\perp} \longrightarrow \mathfrak{g}^*$$

$$(g,\alpha) \longmapsto g\alpha g^{-1}.$$

pf: $\mu(g,\alpha)(x) = H(x)(g,\alpha), \quad x \in \mathfrak{g}.$

$$H(x) = \lambda(\tilde{u}_x).$$

$$\pi : T^*(G/P) \rightarrow G/P$$

$$\begin{array}{ccc} T_{(g,\alpha)}(T^* G/P) & \ni & \tilde{u}_x \\ \downarrow \pi_* & & \downarrow \\ T_g(G/P) & & u_x \end{array} \qquad g\alpha g^{-1} \in T_g^*(G/P).$$

$$H(x)(g,\alpha) = \lambda(\tilde{u}_x)(g,\alpha) = \langle \pi_* \tilde{u}_x , g\alpha g^{-1} \rangle$$

$$= \langle g\alpha g^{-1}, x \rangle$$

$\Rightarrow \mu(g,\alpha) = g\alpha g^{-1}$ $\square$

Hence, the Springer resolution / Grothendieck resolution maps are given by the moment maps.

(V, ω) symplectic vector space,

$W \subseteq V$, $\quad W^{\perp_\omega} := \{ v \in V \mid \omega(v, W) = 0 \}$

<u>Def</u>: W is called

1) isotropic if $\omega|_W = 0$.

2) isotropic if $W^{\perp_\omega}$ is isotropic

3) Lagrangian if $W = W^{\perp_\omega}$

<u>Def</u> A subvariety Z of a symplectic manifold (M, ω) is called an isotropic (resp. coisotropic, Lagrangian) subvariety of M, if for any smooth point $z \in Z$, $T_z Z$ is an isotropic (resp. coisotropic, Lagrangian) subspace of $T_z M$.

Def X a manifold, $Y \subseteq X$ submanifold. The conormal bundle $T^*_Y X \subseteq T^*X$ is the set of all covectors over Y, which annihilate the subbundle $TY \subseteq TX|_Y$.

Prop: $T^*_Y X \subseteq T^*X$ is Lagrangian., and it's stable under the dilations along the fibers of T^*X.

pf: the canonical one-form λ restricts to $T^*_Y X$ is 0 by definition.

$$\omega = d\lambda \Rightarrow \omega|_{T^*_Y X} = 0.$$

$$\dim T^*_Y X = \dim X = \tfrac{1}{2}\dim T^*X \qquad \square$$

Prop (Kashiwara)

$\Lambda \subseteq T^*X$ a closed irreducible algebraic $\mathbb{C}^*$-stable Lagrangian subvariety. $\pi : T^*X \to X$, Y = smooth part of

$\pi(\Lambda)$. Then $\Lambda = \overline{T_Y^* X}$.

pf: $\exists$ natural dilation $\mathbb{C}^*$-action on T^*X.

$Eu :=$ the corresponding vector field.

then $i_{Eu}\omega = \lambda =$ the canonical 1-form.

(locally, $q_1, \cdots, q_n$ local coordinates on X, $p_1, \cdots, p_n$ the dual coordinates on the fiber, then

$$\omega = \sum dp_i \wedge dq_i, \quad \lambda = \sum p_i dq_i, \quad Eu = \sum p_i \frac{d}{dp_i}.$$).

Λ is $\mathbb{C}^*$-stable $\Rightarrow$ Eu is tangent to Λ^{reg} = regular locus of Λ. Since Λ is Lagrangian, for any ξ tangent to Λ^{reg},

$$0 = \omega(Eu, \xi) = \lambda(\xi).$$

$\Rightarrow \lambda|_\Lambda \equiv 0$.

Fix $\alpha \in \Lambda^{reg}$, $y = \pi(\alpha)$. Then by the definition of λ.

α vanishes on the image of the map

$$T_\alpha \Lambda \xrightarrow{\pi_*} T_y Y .$$

Furthermore, Bertini-Sard Lemma implies that $\exists$ a Zariski open dense subset $\Lambda^{generic} \subseteq \Lambda^{reg}$, such that this map is surjective over $\Lambda^{generic}$.

$\Rightarrow \alpha(T_y Y) = 0 \quad \forall \alpha \in \Lambda^{generic}$

$\Rightarrow \alpha \in T_Y^* X .$

$\Rightarrow \Lambda^{generic} \subseteq T_Y^* X.$

$\Rightarrow \Lambda = \overline{\Lambda^{generic}} = \overline{T_Y^* X}$

as both of them are irreducible with the same dim. $\square$

Bertini: the set of critical values (image of the critical points) of a smooth function $f: M \to N$ has Lebesgue measure 0.