

1) general case.

G complex semisimple connected Lie group, $\supseteq B$ Borel subgroup.

$\mathfrak{g} = \text{Lie } G$, $W = \text{Weyl gp.}$

Recall $\pi \in \mathfrak{g}$ is regular if $\dim Z_{\mathfrak{g}}(\pi) = \text{rk } \mathfrak{g}$

semisimple (resp. nilpotent) if $\text{ad } \pi: \mathfrak{g} \rightarrow \mathfrak{g}$ is

semisimple (resp. nilpotent).

Bruhat decomposition $G = \sqcup BwB$.

flag variety $\mathcal{B} := G/B = \{ \text{Borel subgps / subalgs} \}$

$\mathcal{B} \subseteq \text{Gr}(\dim \mathfrak{b}, \mathfrak{g})$.

closed subvariety formed by all solvable Lie subalgs

$\Rightarrow \mathcal{B}$ is a projective variety.

Borel fixed point thm \Rightarrow Plücker embedding

$\mathbb{B} \hookrightarrow \mathbb{P}(V_\lambda)$, λ non-degenerate, dominant weight.

$\mathfrak{b} \mapsto$ the unique line in V_λ fixed by \mathfrak{b} .

Lemma:

$W \rightarrow B \backslash G / B \rightarrow \{ B\text{-orbits on } B \} \rightarrow \{ G\text{-diagonal orbits on } B \times B \}$

$w \mapsto BwB$.

$BgB \mapsto B\text{-orbit of } g.B$

$B\text{-orbit of } \mathfrak{b} \mapsto G \cdot (b_0, \mathfrak{b})$
 $b_0 = \text{Lie } B$.

These maps are bijections.

pf: follows immediately from the Bruhat decomposition \square

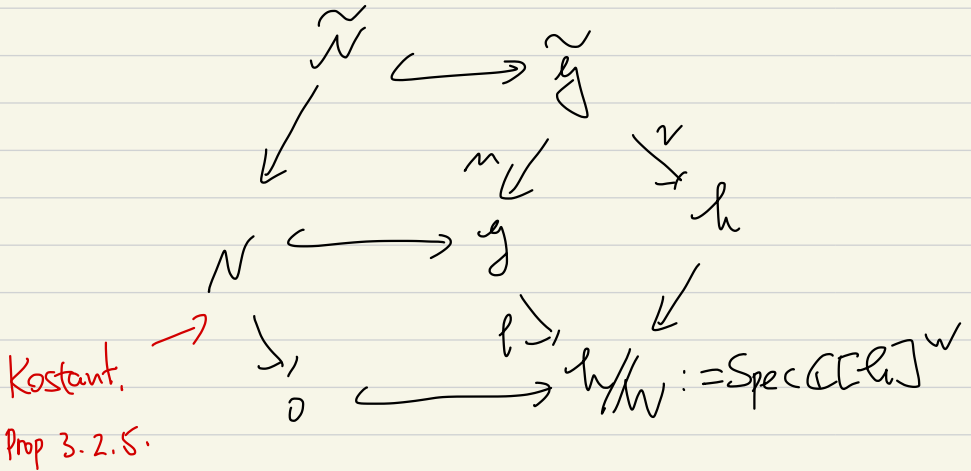
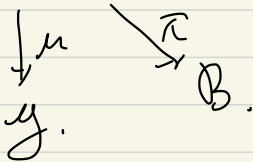
Chevalley restriction thm: $\mathfrak{h} \subseteq \mathfrak{g}$ any Cartan subalgebra.

restriction map gives an isomorphism

$$\mathbb{C}[g]^G \simeq \mathbb{C}[h]^W.$$

Grothendieck's simultaneous resolution

$$\tilde{g} := \{(x, b) \in g \times \mathcal{B} \mid \pi \in b\} \simeq G \times_{\mathcal{B}} \mathcal{B}$$



ρ is induced by $\mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[g]^G \hookrightarrow \mathbb{C}[g]$

$$\nu((x, b)) := x \bmod [b, b] \in \mathfrak{b}/[b, b] \simeq \mathfrak{h}.$$

Thm: 1). The diagrams commute.

2) $\forall x \in \mathfrak{h}$, $\mu: \nu^{-1}(x) \rightarrow \rho^{-1}(x)$ is a resolution of singularities.

3) $\forall x \in \mathfrak{g}^{\text{rs}}$, \exists canonical free W -action on $\mu^{-1}(x)$ making the projection $\tilde{\mathfrak{g}}^{\text{rs}} \rightarrow \mathfrak{g}^{\text{rs}}$ a principle W -bundle.

4) \exists finitely many G -orbits on \mathcal{N} .

5) $\tilde{\mathcal{N}} \simeq T^*\mathcal{B} = G \times_{\mathcal{B}} \mathfrak{N}$, and

$\tilde{\mathcal{N}} \rightarrow \mathcal{N}$ is a resolution of singularities.

(Springer resolution)

2). Moment maps

M a C^∞ -manifold in the \mathbb{R} -case, or smooth alg. variety in the \mathbb{C} -case.

A symplectic structure on M is a non-degenerate regular 2-form ω such that $d\omega = 0$.

$$\underline{\text{Ex}} : M = \mathbb{C}^{2n} \quad (q_1, \dots, q_n, p_1, \dots, p_n)$$

$$\omega = \sum_i dp_i \wedge dq_i$$

$$\underline{\text{Ex}}. M = T^*N, \quad N \text{ smooth.}$$

Construct a one-form λ on M , and set $\omega = d\lambda$.

$$x \in N, \quad \alpha \in T_x^*N, \quad \pi: T^*N \rightarrow N$$

$$\pi_*: T_\alpha(T^*N) \rightarrow T_xN, \quad \gamma \in T_\alpha(T^*N)$$

$$\lambda(\gamma) := \langle \alpha, \pi_*\gamma \rangle \in \mathbb{C}.$$

Ex. $G \curvearrowright \mathfrak{g}^* = \text{the dual of } \mathfrak{g}.$

$\mathcal{O} = \text{a coadjoint orbit } \subseteq \mathfrak{g}^*.$

claim: \mathcal{O} has a natural symplectic structure.

$\alpha \in \mathcal{O}, \quad \mathcal{O} \cong G/G_\alpha, \quad G_\alpha = \text{Stabilizer of } \alpha \text{ in } G$

$T_\alpha \mathcal{O} \cong \mathfrak{g}/\mathfrak{g}_\alpha.$

$\omega_\alpha: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}, \quad (x, y) \mapsto \alpha([x, y]).$

check 1) $\omega_\alpha([x, y], [z, w]) = 0.$

2) $\alpha \mapsto \omega_\alpha$ gives a 2-form ω , show $d\omega = 0$

(M, ω) a symplectic manifold.

$\mathcal{O}(M) \xrightarrow{\cong} \text{vector fields on } M$
 $\downarrow \qquad \qquad \downarrow$
 $f \mapsto \xi_f$

$\omega(\cdot, \xi_f) = df$

define a bracket $\{ \cdot, \cdot \}$ on $\mathcal{O}(M)$

$$\{f, g\} := \omega(\xi_f, \xi_g) = -\xi_g(f)$$

Prop: $\mathcal{O}(M)$ is a Lie alg w.r.t. $\{ \cdot, \cdot \}$, and it

satisfies the Leibniz rule:

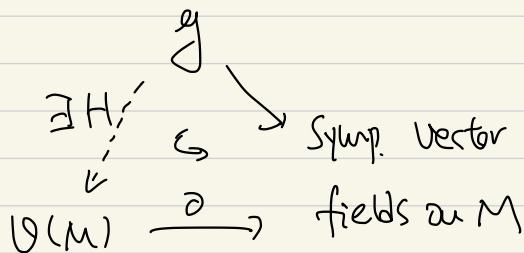
$$\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\}$$

(Poisson alg).

Suppose a Lie Group $G \curvearrowright M$, preserving ω .

$\leadsto \mathfrak{g} \rightarrow$ symplectic vector fields on M .
($\xi, \mathcal{L}_\xi \omega = 0$)

We say the action is Hamiltonian if



Define the moment map $\mu: M \rightarrow \mathfrak{g}^*$

$$m \in M, \quad \chi \in \mathfrak{g}, \quad \mu(m)(\chi) = H_\chi(m).$$

$$M = T^*N, \quad G \curvearrowright N.$$

$\mathfrak{g} \rightarrow$ Vector fields on $N \rightarrow$ Vector fields on M

$$\chi \mapsto U_\chi \mapsto \widehat{U}_\chi$$

Claim: G action on T^*N is Hamiltonian,

$$\text{with } H(\chi) = \lambda(\widehat{U}_\chi) \in \mathcal{O}(M),$$

where $\lambda =$ canonical 1-form on M above.

G Lie group, $P \subseteq G$ subgp.

$$G \curvearrowright G/P, \quad T^*(G/P) \simeq G \times_P (\text{Lie } P)^\perp \subseteq G \times_P \mathfrak{g}^*$$

Prop: the moment map is

$$\mu: T^*(G/P) \cong G \times_{\mathfrak{p}} (\mathfrak{Lie P})^{\perp} \rightarrow \mathfrak{g}^*$$
$$(g, \alpha) \mapsto g \alpha g^{-1}.$$

pf: $\mu(g, \alpha)(\pi) = H(\pi)(g, \alpha)$, $\pi \in \mathfrak{g}$.

$$H(\pi) = \lambda(\tilde{u}_\pi).$$

$$\pi: T^*(G/P) \rightarrow G/P$$

$$\begin{array}{ccc} T_{(g, \alpha)}(T^*(G/P)) & \ni & \tilde{u}_\pi \\ \downarrow \pi_* & & \downarrow \\ T_g(G/P) & & u_\pi \end{array} \quad g \alpha g^{-1} \in T_g^*(G/P)$$

$$\begin{aligned} H(\pi)(g, \alpha) &= \lambda(\tilde{u}_\pi)(g, \alpha) = \langle \pi_* \tilde{u}_\pi, g \alpha g^{-1} \rangle \\ &= \langle g \alpha g^{-1}, \pi \rangle \end{aligned}$$

$$\Rightarrow \mu(g, \alpha) = g \alpha g^{-1}$$

□

Hence, the Springer resolution / Grothendieck resolution maps are given by the moment maps.

(V, ω) symplectic vector space,

$$W \subseteq V, \quad W^{\perp \omega} := \{v \in V \mid \omega(v, w) = 0\}$$

Def: W is called

1) isotropic if $\omega|_W = 0$.

2) isotropic if $W^{\perp \omega}$ is isotropic

3) Lagrangian if $W = W^{\perp \omega}$

Def A subvariety Z of a symplectic manifold (M, ω)

is called an isotropic (resp. coisotropic, Lagrangian)

subvariety of M , if for any smooth point $z \in Z$,

$T_z Z$ is an isotropic (resp. coisotropic, Lagrangian) subspace of $T_z M$.

Def X a manifold, $Y \subseteq X$ submanifold. The conormal bundle $T_Y^*X \subseteq T^*X$ is the set of all covectors over Y , which annihilate the subbundle $TY \subseteq TX|_Y$.

Prop: $T_Y^*X \subseteq T^*X$ is Lagrangian, and it's stable under the dilations along the fibers of T^*X .

pf: the canonical one-form λ restricts to T_Y^*X is 0 by definition.

$$\omega = d\lambda \Rightarrow \omega|_{T_Y^*X} = 0.$$

$$\dim T_Y^*X = \dim X = \frac{1}{2} \dim T^*X \quad \square$$

Prop (Kashiwara)

$\Lambda \subseteq T^*X$ a closed irreducible algebraic \mathbb{G}^* -stable Lagrangian subvariety. $\pi: T^*X \rightarrow X$, $Y = \text{smooth part of}$

$\pi(\Lambda)$. Then $\Lambda = \overline{T_Y^*X}$.

pf: \exists natural dilation \mathbb{C}^* -action on T^*X .

E_Λ : = the corresponding vector field.

then $i_{E_\Lambda} \omega = \lambda$ = the canonical 1-form.

(locally, q_1, \dots, q_n local coordinates on X , p_1, \dots, p_n the dual coordinates on the fiber, then

$$\omega = \sum dp_i \wedge dq_i, \quad \lambda = \sum p_i dq_i, \quad E_\Lambda = \sum p_i \frac{d}{dp_i}.$$

Λ is \mathbb{C}^* -stable $\Rightarrow E_\Lambda$ is tangent to Λ^{reg} = regular locus of Λ . Since Λ is Lagrangian, for any ξ tangent to Λ^{reg} ,

$$0 = \omega(E_\Lambda, \xi) = \lambda(\xi).$$

$$\Rightarrow \lambda|_\Lambda \equiv 0.$$

Fix $\alpha \in \Lambda^{\text{reg}}$, $y = \pi(\alpha)$. Then by the definition of λ .

α vanishes on the image of the map

$$T_x \Lambda \xrightarrow{\pi_*} T_y Y.$$

Furthermore, Bertini-Sard Lemma implies that \exists a

Zariski open dense subset $\Lambda^{\text{generic}} \subseteq \Lambda^{\text{reg}}$, such

that this map is surjective over Λ^{generic} .

$$\Rightarrow \alpha(T_y Y) = 0 \quad \forall \alpha \in \Lambda^{\text{generic}}$$

$$\Rightarrow \alpha \in T_y^* X.$$

$$\Rightarrow \Lambda^{\text{generic}} \subseteq T_y^* X.$$

$$\Rightarrow \Lambda = \overline{\Lambda^{\text{generic}}} = \overline{T_y^* X}$$

as both of them are irreducible with the same dim. \square

Bertini: the set of critical values (image of the critical points) of a smooth

function $f: M \rightarrow N$ has Lebesgue measure 0.