Thm (Lefschetz-type formula).

$f: X \to Y$ proper, A-equivariant, X, Y smooth

Assume $a \in A$ is both X and Y regular. then

$$\begin{tikzcd} K^A(X) \arrow[r, "f_*"] \arrow[d, "res_a"'] \arrow[dr, phantom, "\circlearrowright"] & K^A(Y) \arrow[d, "res_a"] \\ K_{\mathbb{C}}(X^A) \arrow[r, "f_*"'] & K_{\mathbb{C}}(X^A) \end{tikzcd}$$

Remarks 1)

In literacture, people usually don't choose $a \in A$.

Instead, we consider $f: X \to pt$, X compact

$$f_* \mathcal{F} = \sum (-1)^i H^i(X, \mathcal{F}) \in K^A(pt) = R(A)$$

Regard this as a function on A, insteady of evaluating

at $a \in A$.

Then we have

$$\boxed{f_* \mathcal{F} = \sum (-1)^i H^i(X, \mathcal{F}) = \overline{\sum} (-1)^i H^i\left(X^A, \frac{\mathcal{F}}{\sum (-1)^i \Lambda^i N^\vee}\right).}$$

In particular, if $X^A = \{p_i\}$ consists of finitely many pts

$$\boxed{f_* \mathcal{F} = \underbrace{\sum (-1)^i H^i(X, \mathcal{F})}_{\chi(X,\mathcal{F})} = \sum_{p_i} \frac{\mathcal{F}|_{p_i}}{\sum (-1)^i \Lambda^i (T^*_{p_i} X)} \in K^A(pt)}$$

2) Recall $\lambda_A \cdot K^A(M^A)_a \xrightarrow{\sim} K^A(M^A)_a$ for M-regular a

if we use the notation $K^A(M)_{loc}$, then

$$\lambda_A \cdot K^A(M^A)_{loc} \xrightarrow{\sim} K^A(M)_{loc},$$

Thus, λ_A^{-1} exists

Hence $\forall\ \mathcal{F} \in K^A(M)$, let $(i_*)^{-1}\mathcal{F} = \mathcal{G} \in K^A(M^A)_{loc}$,

Such $\mathcal{G}$ exists since $i_* : K^A(M^A)_{loc} \xrightarrow{\sim} K^A(M)_{loc}$.

then $i^* i_* \mathcal{G} = \lambda_A \otimes \mathcal{G}$

$$\Rightarrow \quad i_* \frac{i^* \mathcal{F}}{\lambda_A} = \mathcal{F}.$$

If we write $M^A = \sqcup F_j$, then

$$\boxed{\mathcal{F} = \sum_j (i_j)_* \frac{\mathcal{F}|_{F_j}}{\sum_k (-1)^k \cdot \Lambda^k (N^\vee_{F_j/M})}}$$

In particular, even if M is not compact,

but M^A is compact, we can still define $\forall \mathcal{F} \in K^A(M)$

$$f_* \mathcal{F} := \sum (f_j)_* \frac{\mathcal{F}|_{F_j}}{\sum (-1)^k \Lambda^k (N^\vee_{F_j/M})}.$$

where
$$\begin{tikzcd} F_j \arrow[r, hook, "i_j"] \arrow[dr, "f_j"'] & M \arrow[d, "f"] \\ & pt \end{tikzcd}$$

This is used widely in the literatures, for example,

Gromov-Witten theory.

Example: $X = \mathbb{P}^1 \quad A = \left\{ \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} \right\} \circlearrowright \mathbb{P}^1$

Let's compute $\chi(X, \mathcal{O}_{\mathbb{P}^1}(d)) \in R(A)$.

Firstly, let's assume $d \geq 0$, then

$$\chi(X, \mathcal{O}_{\mathbb{P}^1}(d)) = H^0(\mathcal{O}_{\mathbb{P}^1}(d)) - \overset{0}{\overset{\shortparallel}{H^1(\mathcal{O}_{\mathbb{P}^1}(d))}}$$

$$= \mathrm{Span}_{\mathbb{C}}\{x^d, x^{d-1}y, \cdots, xy^{d-1}, y^d\}.$$

Thus, as a rep of A, the character is

$$\chi(X, \mathcal{O}_{\mathbb{P}^1}(d)) = z^{-d} + z^{-d+2} + \cdots + z^{d-2} + z^d$$

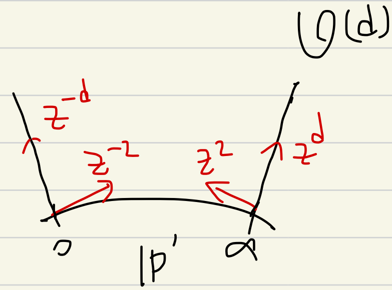
$$= \frac{z^{d+1} - z^{-d-1}}{z - z^{-1}}$$

Exercise. Compute it for $d < 0$,

and check we get the same formula.

Secondly, let's do it using localization

$$(\mathbb{P}^1)^A = \{ \underset{\substack{\| \\ 0}}{[1:0]}, \underset{\substack{\| \\ \infty}}{[0:1]} \}$$



$T_0\mathbb{P}^1 = \mathrm{Hom}(\mathbb{C}(1:0), \mathbb{C}^2/\mathbb{C}(1:0))$

has character z^{-2}

$T_\infty \mathbb{P}^1$ has character z^2

Recall the taut line bundle $\mathcal{O}(-1)|_{\mathbb{C}v} = \mathbb{C}v$.

$\Rightarrow$ $\mathcal{O}(-1)|_0$ has char. z,

$\mathcal{O}(-1)|_\infty$ has char. z^{-1}.

$\Rightarrow$ $\mathcal{O}(d)|_0$ has char. z^{-d}

$\mathcal{O}(d)|_\infty$ has char. z^{d}.

Thus, $\chi(\mathbb{P}^1, \mathcal{O}(d)) = \dfrac{z^{-d}}{1-z^2} + \dfrac{z^d}{1-z^{-2}} = \dfrac{z^{d+1} - z^{-d-1}}{z - z^{-1}}$.

§. Equivariant K-theory of the flag variety

Some classical results

<u>Thm</u>: 1) $R(G) \simeq R(T)^W$

2) if G is simply connected,

$R(T)$ is a free $R(T)^W$-mod, and

$R(T) \simeq R(T)^W \otimes_{\mathbb{Z}} \mathbb{Z}[W]$

<u>Cor</u> 1) $K^G(G/B) \simeq R(T)$

2) If G is simply connected, $K^G(G/B)$ is a free $R(G)$-mod.

Pf: 1) $K^G(G/B) \simeq K^B(pt) \simeq R(B) \simeq R(T)$.

2) follows from the above results. $\square$

Conventions: From now on, G semisimple, simply-conn.

$X^*(T) := \mathrm{Hom}(T, \mathbb{C}^*) =$ weight lattice.

$R^+ =$ the set of weights in the natural T-action on $\mathfrak{g}/\mathfrak{b}$

(this is opposite to the usual one; it's called geometric in [CG])

$\lambda \in X^*(T)$ is called dominant if $\lambda(\alpha^\vee) \geq 0$, $\forall \alpha \in R^+$

G-equiv. line bundles on $G/B =: \mathcal{B}$.

$\lambda \in X^*(T)$, $\quad B \to B/[B,B] = T \xrightarrow{\lambda} GL(\mathbb{C}_\lambda)$

$$\begin{array}{c} \mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda \\ \downarrow \\ G/B \end{array}$$

i.e. let $p: G \to G/B$, $\quad \forall\, U \overset{\text{open}}{\subseteq} G/B$

$$\Gamma(U, \mathcal{L}_\lambda) = \left\{ \tilde{s} : p^{-1}(U) \to \mathbb{C} \;\middle|\; \begin{array}{l} \tilde{s}(g \cdot b) = \lambda(b)^{-1} \tilde{s}(g),\ \forall g \in G,\ b \in B, \\ gb, g \in p^{-1}(U) \end{array} \right\}$$

Conversely, for any G-equiv. line bundle $\mathcal{L}$ on G/B,

$\mathcal{L}|_B$ is a one-dim'l B-mod.

Hence factor through $B/[B,B] = T$, $\mathcal{L}|_B \simeq \mathbb{C}_\lambda \in \mathrm{Rep}(B)$.

$\Rightarrow \mathcal{L} \simeq \mathcal{L}_\lambda$.

<u>Prop</u>. $K^G(G/B) \simeq R(T)$ is given by $\mathcal{L}_\lambda \mapsfrom \lambda \in X^*(T)$.

For $\lambda \in X^*(T)$ anti-dominant. (dominant in the usual choice)

$\rightsquigarrow$ f.d. irrep V_λ of G with highest weight λ

For any Borel $B' \leq G$, $\exists!$ B'-stable line $l_{B'} \leq V_\lambda$ on which B'

acts via $B' \to B'/[B',B'] \xrightarrow{\sim} B/[B,B] \to GL(\mathbb{C}_\lambda)$.

$\rightsquigarrow \phi : \mathcal{B} \to \mathbb{P}(V_\lambda)$, $B' \mapsto l_{B'}$.

then $\phi^* \mathcal{O}(-1) = \mathcal{L}_\lambda$.

Hence, $\mathcal{L}_\lambda$ is positive when λ is dominant.

(reason for the geometric choice in [CG])

Weyl character

λ dominant, $G \curvearrowright H^0(\mathcal{B}, \mathcal{L}_\lambda)$ by $(x.\tilde{s})(g) = \tilde{s}(x^{-1}g)$, $x, g \in G$.

$w_0 \in W$, the longest element.

Thm: For dominant λ, $H^0(\mathcal{B}, \mathcal{L}_\lambda) \simeq V_{w_0(\lambda)}$.

pf: $B = TU$, $U w_0 B/B \subseteq \mathcal{B}$. Zariski open dense.

$\Rightarrow$ any $\tilde{s} \in H^0(\mathcal{L}_\lambda)$ is determined by its restriction to Uw_0B/B.

$\tilde{s}(uw_0b) = \lambda(b)^{-1} \cdot \tilde{s}(uw_0)$, $u \in U$, $b \in B$

Thus, there is at most one such function which is left U-inv.

Moreover, if $\tilde{s}$ is a such function,

then for $t \in T$, $u \in U$

$$(t\tilde{s})(uw_0) = \tilde{s}(t^{-1}uw_0) = \tilde{s}(\overbrace{t^{-1}ut}^{\substack{U \\ \in}}\, t^{-1}w_0) = \tilde{s}(t^{-1}w_0)$$

$$= \tilde{s}(w_0 w_0^{-1} t^{-1} w_0) = \lambda(w_0^{-1} t w_0)\, \tilde{s}(w_0) = (w_0\lambda)(t) \cdot \tilde{s}(w_0).$$

Thus, $\tilde{s}$ is a weight vector for the left B-action, with wt $w_0\lambda$.

Hence, if $H^0(\mathcal{B}, \mathcal{L}_\lambda) \neq 0$, highest weight theory shows

$H^0(\mathcal{B}, \mathcal{L}_\lambda)$ is irreducible, and $\simeq V_{w_0\lambda}$

Let's prove $H^0(\mathcal{B}, \mathcal{L}_\lambda) \neq 0$.

claim: $H^i(\mathcal{B}, \mathcal{L}_\lambda) = 0 \quad \forall i > 0$

(if λ is non-degenerate, then $\mathcal{B} \hookrightarrow \mathbb{P}(V_\lambda)$ is the Plücker

embedding, and $\mathcal{L}_\lambda$ is ample,

$H^i(\mathcal{L}_\lambda) = 0 \ \forall i > 0$ follows from Kodaira vanishing

other cases need more work).

Let's assume $H^i(\mathcal{B}, \mathcal{L}_\lambda) = 0 \quad \forall i > 0$.

$\Rightarrow \chi(\mathcal{B}, \mathcal{L}_\lambda) = \sum (-1)^i H^i(\mathcal{B}, \mathcal{L}_\lambda) = H^0(\mathcal{B}, \mathcal{L}_\lambda) \in R(T)$.

On the other hand, we can compute LHS. via Localization.

$(\mathcal{B})^T \longleftrightarrow W$

$w(b) \longleftarrow w$

$$\chi(\mathcal{B}, \mathcal{L}_\lambda) = \sum_{w \in W} \frac{\mathcal{L}_\lambda|_{wb}}{\sum (-1)^i \Lambda^i (T^*_{wb}\mathcal{B})}$$

weights $(T_{wb}\mathcal{B})$
$= \{ w\alpha \mid \alpha > 0 \}$

$$= \sum_{w \in W} \frac{e^{w\lambda}}{\prod_{\alpha>0} (1 - e^{-w\alpha})}$$

$\alpha > 0$
$\Rightarrow w_0\alpha < 0$

$$= \sum_{w} \frac{e^{ww_0\lambda}}{\prod_{\alpha>0} (1 - e^{+w\alpha})}$$

$2\rho = \sum_{\alpha>0} \alpha$

$$= \sum_{w} \frac{e^{w(w_0\lambda - \rho)}}{\prod_{\alpha>0} (e^{-w\alpha/2} - e^{w\alpha/2})}$$

$\#\{ w\alpha < 0 \mid \alpha > 0 \}$
$= \ell(w)$

$$= \sum_{w} (-1)^{\ell(w)} \frac{e^{w(w_0\lambda - \rho)}}{\prod_{\alpha>0} (e^{-\alpha/2} - e^{\alpha/2})}$$

$$= \operatorname{char} V_{w_0\lambda}.$$

$\square$

Künneth formula for $\mathcal{B}$.

Prop: The Künneth formula holds for $\mathcal{B}$.

pf: use the following criterion.

$K^G(\mathcal{B})$ and $K^G(\mathcal{B}\times\mathcal{B})$ are f.g projective $R(G)$-mods

$\mathrm{rk}\, K^G(\mathcal{B}\times\mathcal{B}) = (\mathrm{rk}\, K^G(\mathcal{B}))^2$,

$\langle -,-\rangle : K^G(X)\times K^G(X) \to R(G)$ is non-degenerate.

$K^G(\mathcal{B}) \simeq R(T)$ is a free $R(G)\simeq R(T)^W$-mod of $\mathrm{rk} = |W|$

the case for $\mathcal{B}\times\mathcal{B} = \sqcup Y_w$ follows from the cellular fibration lemma (Lemma 5.5.1 in [CG])

The non-degeneracy of $\langle -,-\rangle$. use a basis $\{e_y\}_{y\in W}$ of the free $R(T)^W$-mod $R(T)$, constructed by Steinberg, s.t.

$$\det A = \Delta^{|W|/2},$$

where $A = (w(e_y))$, $\Delta = \prod_{\alpha>0}(e^{\alpha/2} - e^{-\alpha/2})$ = the Weyl denominator.

Under the isomorphism $K^G(\mathcal{B}) \simeq R(T)$,

$$\langle -,- \rangle : R(T) \times R(T) \to R(G)$$

$$(P, Q) \mapsto \Delta^{-1} \cdot \sum_{w \in W} (-1)^{l(w)} w(PQ) \cdot e^{w(\rho)}.$$

($\Leftarrow$ localization)

Hence, $\langle e_y, e_{y'} \rangle = \Delta^{-1} \sum_w (-1)^{l(w)} w(e_y)\, w(e^\rho) \cdot w(e_{y'})$

$$= (A \cdot D A^t)_{y,y'}$$

where $A_{y,w} = w(e_y)$, $D = \mathrm{diag}\left(\Delta^{-1} (-1)^{l(w)} w(e^\rho)\right)$

$$\Rightarrow \det(\langle e_y, e_{y'} \rangle) = (\det A)^2 \cdot \det D$$

$$= \Delta^{|W|} \Delta^{-l(w)} (-1)^{\sum_w l(w)} e^{\sum_w w(\rho)}$$

$$= \pm 1,$$

the last equality follows from the fact that

$\sum_w w(\rho) \in (X^*(T))^W \Rightarrow \sum_w w(\rho) = 0.$

$\square$

For any G-variety X, $W = N_G(T)/T$

$N_G(T) \circlearrowright X \rightsquigarrow W \circlearrowright K_T(X)$

Thm. 1) $R(T) \otimes_{R(G)} K^G(X) \simeq K^T(X)$.

2) $K^G(X) \simeq (K^T(X))^W$.

Pf. $G \times_B X \xrightarrow{\sim} G/B \times X$

$(g, x) \longmapsto (gB, gx)$

$$\rightsquigarrow K^T(X) \underset{B = TU}{\simeq} K^B(X) \underset{\text{induction}}{\simeq} K^G(G \times_B X) \simeq K^G(\mathcal{B} \times X)$$

$$\underset{\text{Kunneth}}{=} K^G(\mathcal{B}) \otimes_{K^G(pt)} K^G(X) \simeq R(T) \otimes_{R(G)} K^G(X),$$

$R(T) \simeq \mathbb{Z}[W] \otimes_{\mathbb{Z}} R(G)$

$\Rightarrow K^T(X) \simeq \mathbb{Z}[W] \otimes_{\mathbb{Z}} K^G(X)$ $\square$