§. Local Langlands conj.

K a non-arch local field, ($K = \mathbb{F}_q((t))$ or finite ext of $\mathbb{Q}_p$)

$\mathcal{O} \subseteq K$ ring of integers. $k = \mathbb{F}_q$ = residue field.

G split reductive group $/K$, $\hat{G}$ = dual group.

Local Langlands correspondence (LLC):

$$\left\{ \begin{matrix} \text{irreducible admissable} \\ \text{reps of } G(K) \text{ on } \mathbb{C}\text{-vector spaces} \end{matrix} \right\} \Big/ \simeq \quad \overset{\text{finite 1}}{\longleftrightarrow} \quad \left\{ \begin{matrix} F\text{-semisimple Weil-Deligne} \\ \text{reps in } \hat{G}(\mathbb{C}) \end{matrix} \right\} \Big/ \simeq$$

- A rep V of $G(K)$ is called admissable if $\forall$ open subgp $U \subseteq G(K)$, $\dim V^U < \infty$.

-

$$\begin{tikzcd} I_{\bar{K}/K} \arrow[r, hook] \arrow[d, equal] & \mathrm{Gal}(\bar{K}/K) \arrow[r, two heads, "\varphi"] & \mathrm{Gal}(\bar{k}/k) = \hat{\mathbb{Z}} \\ I_{\bar{K}/K} \arrow[r, hook] & W_K := \varphi^{-1}(\mathbb{Z}) \arrow[u, phantom, "\cup" rotate=90] \arrow[r, two heads] & \langle \mathrm{Frob} \rangle = \mathbb{Z} \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

$\uparrow$ Weil group

- A Weil-Deligne rep in $\hat{G}$ is a pair (ρ, x), where
  - $\rho: W_K \to \hat{G}(\mathbb{C})$ is a continuous group homomorphism
  - $x \in \mathrm{Lie}\,\hat{G}(\mathbb{C})$ is nilpotent,

such that $\rho(g)x\rho(g)^{-1} = |g|x \quad \forall g \in W_K \quad (|\mathrm{Frob}| = |k| = q)$

(this implies x is nilpotent).

- (ρ, x) is F-semisimple if $\rho(\mathrm{Frob}) \in \hat{G}(\mathbb{C})$ is semisimple.

---

unramified case:

$$\left\{ \begin{array}{l} \text{irr. admissable unramified reps of } G(K), \\ \text{i.e. reps that admit a non-zero } G(\mathcal{O})\text{-fixed vector} \end{array} \right\} \Big/ \simeq \qquad (*)$$

$$\updownarrow \; 1{:}1$$

$$\left\{ \begin{array}{l} \text{unramified semisimple Weil-Deligne reps,} \\ \text{i.e. reps factor through } W_K \twoheadrightarrow \mathbb{Z} \to \check{G}(\mathbb{C}) \\ \text{and } x = 0 \end{array} \right\} \Big/ \simeq \qquad (**)$$

· Spherical Hecke alg $\mathbb{C}_c[G(\mathcal{O})\backslash G(K)/G(\mathcal{O})] =: \mathcal{H}_{sph}$,

alg. structure is given by convolution

$$(f_1 * f_2)(g) = \int_{G(K)} f_1(gx^{-1}) f_2(x)\,dx,$$

and it is a commutative alg.

· For any rep V in (*),

$\mathcal{H}_{sph} \curvearrowright V^{G(\mathcal{O})}$ by

$$f\, v = \int_{G(K)} f(g)\, g.v\, dg.$$

Moreover, this gives a bijection between

$$(*) \overset{1:1}{\longleftrightarrow} \{\text{irreducible modules for } \mathcal{H}_{sph}\}/\simeq$$

<u>Thm</u> (Satake isomorphism)

$$\mathcal{H}_{sph} \xrightarrow{\sim} R(\hat{G}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[\hat{T}]^W.$$

Thus, $\{\text{irreducible reps of } \mathcal{H}_{sph}\}/\simeq \xleftrightarrow{1:1} \left\{\begin{array}{l}\text{semisimple conjugacy} \\ \text{classes in } \hat{G}(\mathbb{C})\end{array}\right\}$

$$\Big\updownarrow 1:1$$

(**)

Thus, the Satake isomorphism $\Rightarrow$ unramified LLC.

Rmk: categorification of the Satake isomorphism is given by the geometric Satake equivalence.

The tamely ramified with unipotent monodromy (TRUM) case.

LLC:

$$\left\{\begin{array}{l}\text{TRUM reps of } G(K) \\ \text{i.e. reps. admit a non-zero} \\ \text{Iwahori fixed vector}\end{array}\right\}/\simeq \xleftrightarrow{\text{finite:1}} \left\{\begin{array}{l}\text{TRUM Weil-Deligne reps. i.e.} \\ \text{reps factor through} \\ W_K \twoheadrightarrow \mathbb{Z} \to \hat{G}(\mathbb{C}),\ x \text{ arbitrary}\end{array}\right\}/\simeq$$

Here, Iwahori subgp

$$\begin{array}{ccc} I & \hookrightarrow & G(\mathcal{O}) \\ \downarrow & & \downarrow \\ B(k) & \hookrightarrow & G(k) \end{array}$$

Iwahori-Hecke alg. $\mathcal{H}_{\mathrm{aff}} := \mathbb{C}_c[I \backslash G(K)/I]$

As before, LHS $\overset{1:1}{\longleftrightarrow}$ $\{$finite dim'l irreps of $\mathcal{H}_{\mathrm{aff}}\}$

RHS $\overset{1:1}{\longleftrightarrow}$ $\{(s,x) \in \hat{G}(\mathbb{C}) \times \mathcal{N} \mid s \text{ semisimple}, sxs^{-1} = qx\} / \mathrm{conj}$

$s \in \hat{G}(\mathbb{C})$ is the image of Frob,

$\mathcal{N} \subseteq \mathrm{Lie}\,\hat{G}(\mathbb{C})$ is the nilpotent cone.

Hence, LLC becomes

<u>Deligne-Langlands conj</u>

$$\{\text{finite dim'l irreps of } \mathcal{H}_{\mathrm{aff}}\}/\simeq \ \overset{\text{finite}:1}{\longleftrightarrow} \ \left\{(s,x) \in \hat{G}(\mathbb{C}) \times \mathcal{N} \;\middle|\; \begin{matrix} s \text{ s.s.} \\ sxs^{-1} = qx \end{matrix}\right\} / \mathrm{conj}$$

<u>Refined version by Lusztig.</u>

Add more data on the RHS:

+ irr. $\hat{G}(\mathbb{C})$-equiv. local system on the conjugacy classes of (s,x)

and this is equiv. to the reps of $C(s,x)$ = the component group for the simultaneous centralizer of both s and x

<u>Deligne - Langlands - Lusztig.</u>

$$\{\text{finite dim'l irrep of } \mathcal{H}_{aff}\}/_{\simeq} \xleftrightarrow{1:1} \left\{(s,x,\psi) \,\middle|\, \begin{matrix} s \in \hat{G}(\mathbb{C}) \text{ s.s.}, \ x \in \mathcal{N}, \\ sxs^{-1} = qx, \ \psi \in \hat{C}(s,x) \end{matrix}\right\}/_{\text{conj}}$$

This is proved by Kazhdan-Lusztig and Ginzburg.

The goal of the rest of this course is to explain the proof.

1st step:

$$\mathbb{C}_c[I\backslash G(K)/I] \overset{\text{Iwahori-Matsumoto}}{\simeq} \text{Hecke alg for } W_{aff} \underset{\text{Bernstein}}{\simeq} H_{aff} \overset{\text{Kazhdan-Lusztig, Ginzburg}}{\simeq} K^{\hat{G}(\mathbb{C}) \times \mathbb{C}^*} \ (\text{Steinberg})$$

2nd step. use sheaf-methods to classify the irreps of $K^{\hat{G}(\mathbb{C}) \times \mathbb{C}^*}$ (Steinberg)

Since we are going to focus on $\hat{G}(\mathbb{C})$, we will use G for it from now on

§ Affine Hecke alg.

G simply connected, semisimple alg group$/\mathbb{C}$. $\quad T \overset{\text{max torus}}{\subseteq} G$

$P = \mathrm{Hom}_{alg}(T, \mathbb{C}^*) =$ weight lattice.

$W_{aff} := W \ltimes P$ (extended) affine Weyl group

<u>Def</u>: The affine Hecke alg $\mathbb{H}$ is a free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{e^\lambda T_w \mid w \in W, \lambda \in P\}$, s.t.

a) $(T_s+1)(T_s-q)=0$ if s is a simple reflection

b) $T_y T_w = T_{yw}$ if $l(yw) = l(y) + l(w)$

c) $e^\lambda e^\mu = e^{\lambda+\mu}$

d) α_s simple, if $\langle \lambda, \alpha_s^\vee \rangle = 0$, $T_{s_\alpha} e^\lambda = e^\lambda T_{s_\alpha}$

e) α_s simple, if $\langle \lambda, \alpha_s^\vee \rangle = 1$, $T_{s_\alpha} e^{s_\alpha(\lambda)} T_{s_\alpha} = q\, e^\lambda$.

Rmks:

1) a) + b) $\Rightarrow \{T_w \mid w \in W\} = H_W =$ the finite Hecke alg.

2) d) + e) $\Leftrightarrow T_{s_\alpha} e^{s_\alpha(\lambda)} - e^\lambda T_{s_\alpha} = (1-q) \dfrac{e^\lambda - e^{s_\alpha(\lambda)}}{1-e^{-\alpha}}$

3). c) $\Rightarrow$ $R(T)[q,q^{-1}] \hookrightarrow \mathbb{H}$.

4) $\mathbb{H} \simeq R(T)[q,q^{-1}] \otimes_{\mathbb{Z}[q,q^{-1}]} H_W$ as $\mathbb{Z}[q,q^{-1}]$-modules

5) In fact, $\mathbb{H}$ defined above is the extended affine Hecke alg. for the Langlands dual group.

$\forall \lambda \in P$, let $z(e^\lambda) := \sum_{\lambda' \in W\cdot\lambda} e^{\lambda'}$.

<u>Thm</u> (Bernstein)

The center $Z(\mathbb{H})$ is a free $\mathbb{Z}[q,q^{-1}]$-mod with basis $\{z(e^\lambda) \mid \lambda \in \underset{\text{dominant}}{P_+}\}$,

and $Z(\mathbb{H}) \simeq R(T)^W [q,q^{-1}]$.

§ Equivariant K-theory and Hecke alg.

ref. Lusztig, equivariant K-theory and representations of Hecke alg.

Recall $K^{\mathbb{C}^*}(pt) = \mathbb{Z}[q^{\pm 1}]$.

It's Lusztig who first realized that this q is related to the q in the Hecke alg. Thus, we should be able to use equiv. K-theory to study Hecke algs.

Lusztig defined an action of the affine Hecke alg. on $K^G(G/B)$

Firstly, $K^G(G/B) \simeq K^B(pt) = R(T)$, it has a $\mathbb{Z}$-basis $e_\lambda = G\times_B \mathbb{C}_\lambda$

For any simple reflection $s_\alpha \in W$, let $P_s := B \cup B s_\alpha B$

consider
$$\begin{array}{ccc} P_s/B & \hookrightarrow & G/B \\ & & \downarrow \pi_s \\ & & G/P_s \end{array}$$
a $P_s/B \simeq \mathbb{P}^1$ fibration.

$\Omega'_s :=$ relative cotangent sheaf of π_s

the action of $\mathbb{H}$ on $K^G(G/B)$ is defined as follows.

$\forall [\mathcal{F}] \in K^G(G/B)$,

$$T_s([\mathcal{F}]) = \pi_s^* \pi_{s*}[\mathcal{F}] - [\mathcal{F}] - q\pi_s^* \pi_{s*}([\mathcal{F}] \otimes [\Omega_s^1])$$

$\forall \lambda \in X^*(T)$,

$$e^\lambda([\mathcal{F}]) := [\mathcal{F}] \otimes [\mathcal{L}_\lambda^*]$$

Thm. The above defines an action of $\mathbb{H}$ on $K^{G\times\mathbb{C}^*}(G/B)$, where $\mathbb{C}^*$ acts trivially on G/B.

pf: Let's compute these operators under the isomorphism

$$K^{G\times\mathbb{C}^*}(G/B) \simeq \mathbb{Z}[q,q^{-1}][X^*(T)].$$

$e^\lambda \in \mathbb{H} \rightsquigarrow$ multiplication by $e^{-\lambda}$.

Let's compute T_s.

Take $[\mathcal{F}] = [\mathcal{L}_\lambda]$, what's $\pi_s^* \pi_{s*}[\mathcal{L}_\lambda]$?

The isomorphism $K^G(G/B) \simeq R(T)$ is given by

$$[f] \mapsto [f]|_B$$
$$\shortparallel$$
the pullback $B \hookrightarrow G/B$.

Thus, we need to compute $\pi_s^* \pi_{s*}[\mathcal{L}_\lambda]|_B$.

$$\begin{tikzcd} B \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & G/B \arrow[d, "\pi_s"] \\ P_s \arrow[r, hook] & G/P_s \end{tikzcd} \quad \rightsquigarrow \quad \pi_s^* \pi_{s*}[\mathcal{L}_\lambda]|_B$$

$$= i^* \pi_s^* \pi_{s*}[\mathcal{L}_\lambda]$$

$$= \pi_{s*}[\mathcal{L}_\lambda]|_{P_s}$$

$\Omega_s^1|_B$ has weight $-\alpha$

localization $\Rightarrow$

fiber $\pi_s^{-1}(P_s)$ has two fixed points $\{B, s_\alpha B\}$.

$$= \frac{[\mathcal{L}_\lambda]|_B}{\lambda[\Omega_s^1|_B]} + \frac{[\mathcal{L}_\lambda]|_{s_\alpha B}}{\lambda[\Omega_s^1|_{s_\alpha B}]}$$

$$= \frac{e^\lambda}{1-e^{-\alpha}} + \frac{e^{s_\alpha \lambda}}{1-e^{+\alpha}},$$ it lies in $R(T)$.

Thus, $T_s \cdot = \pi_s^* \pi_{s*} - id - q \cdot \pi_s^* \pi_{s*}(\Omega_s^1 \otimes -)$

gives the operator on $R(T)[q, q^{-1}]$

$$T_s(e^\lambda) = \frac{e^\lambda}{1-e^{-\alpha}} + \frac{e^{s_\alpha \lambda}}{1-e^{+\alpha}} - e^\lambda - q\left(\frac{e^{\lambda-\alpha}}{1-e^{-\alpha}} + \frac{e^{s_\alpha(\lambda-\alpha)}}{1-e^{+\alpha}}\right)$$

$$= \frac{e^{\lambda} - e^{s_\alpha \lambda}}{e^{+\alpha} - 1} - q \frac{e^{\lambda} - e^{s_\alpha \lambda + \alpha}}{e^{+\alpha} - 1}$$

<u>Rmk</u>: 1) $q=1$, $T_s(e^{\lambda})|_{q=1} = s_\alpha(e^{\lambda})$

2) $q=0$, Demazure operator.

3) This is called Demazure-Lusztig operator.

Now we only need to check all the relations in the affine Hecke alg.

the only non-trivial one is the braid relation, i.e we need to check for all the rank 2 root systems. Lusztig checked it by direct computation.

$\square$

§. Main results

$$\begin{array}{l}\tilde{\mathcal{N}} \\ \downarrow \mu \\ \mathcal{N}\end{array} \quad \text{Springer resolution.} \qquad G\times\mathbb{C}^* \curvearrowright \mathcal{N}$$

$$(g,z)\,x = z^{-1}\cdot gxg^{-1}, \qquad g\in G,\ z\in\mathbb{C}^*,\ x\in\mathcal{N}.$$

$$G\times\mathbb{C}^* \curvearrowright \tilde{\mathcal{N}} \simeq T^*\mathcal{B}$$

$$(g,z)\cdot(x,b) = (z^{-1}gxg^{-1},\ gbg^{-1}). \qquad b\in\mathcal{B},\ x\in T_b^*\mathcal{B}\simeq\mathfrak{n}$$

$\mathbb{C}^*$-acts trivially on $\mathcal{B}$.

$$G\times\mathbb{C}^* \curvearrowright Z := T^*\mathcal{B}\times_{\mathcal{N}} T^*\mathcal{B}$$

q^{-1} = identity rep of $\mathbb{C}^*$, $\mathbb{C}^*$ acts on fibers of $T^*\mathcal{B}$ by char. q.

(note this is opposite to the choice in [CG] in 7.2.3).

$$Z_\Delta := T^*_{\mathcal{B}_\Delta}(\mathcal{B}\times\mathcal{B}) \subseteq Z$$

$$\begin{array}{l} Z_\Delta \\ \downarrow \\ \mathcal{B}_\Delta\end{array} \qquad K^{G\times\mathbb{C}^*}(Z_\Delta)\simeq K^{G\times\mathbb{C}^*}(\mathcal{B}_\Delta)\simeq R(T)[q,q^{-1}].$$

<u>Thm</u>: $\exists$ natural alg isomorphism $K^{G\times\mathbb{C}^*}(Z)\simeq\mathbb{H}$,

s.t.

$$\begin{array}{ccc} K^{G\times\mathbb{C}^*}(Z_\Delta) & \hookrightarrow & K^{G\times\mathbb{C}^*}(Z) \\ \downarrow\wr & \circlearrowright & \downarrow\wr \\ R(T)[q,q^{-1}] & \hookrightarrow & \mathbb{H} \end{array}$$

<u>Rmks</u>. 1). $K^{G\times\mathbb{C}^*}(pt) \simeq R(T)^W[q,q^{-1}] \subset R(T)[q,q^{-1}] \simeq K^{G\times\mathbb{C}^*}(Z_\Delta)$

$\hookrightarrow K^{G\times\mathbb{C}^*}(Z) \simeq \mathbb{H}$ is the Center of $\mathbb{H}$.

2) forget the $\mathbb{C}^*$-action, we get

$$\begin{tikzcd} K^G(Z_\Delta) \arrow[r, hook] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & K^G(Z) \arrow[d, "\wr"] \\ R(T) \arrow[r, hook] & \mathbb{Z}[W_{aff}] \end{tikzcd}$$

The isomorphism $K^G(Z) \simeq \mathbb{Z}[W_{aff}]$ can be proved similarly as the isomorphism $H(Z) \simeq \mathbb{C}[W]$, via using the Grothendieck-Springer resolution i.e for $h \in \mathfrak{h}^{reg}$, we consider

$$\Lambda_w^h = \text{graph of } (\tilde{\mathfrak{g}}^h \xrightarrow{w} \tilde{\mathfrak{g}}^{w(h)}).$$

However, since the $\mathbb{C}^*$-action doesn't preserve $\tilde{\mathfrak{g}}^h$, this argument doesn't work for $K^{G\times\mathbb{C}^*}(Z)$

For more details, see Section 7.3 in [CG].