§. $SL_2(\mathbb{C})$ - case

$\mathcal{B} \simeq \mathbb{P}^1 =: \mathbb{P}$, $\mathbb{P} \times \mathbb{P} = \mathbb{P}_\Delta \sqcup Y$ two diagonal G-orbits.

$Z = T^*\mathbb{P} \times_{\mathcal{N}} T^*\mathbb{P} = Z_\Delta \cup Z_Y$,

where $Z_\Delta := T^*_{\mathbb{P}_\Delta}(\mathbb{P}\times\mathbb{P})$, $Z_Y := T^*_{\overline{Y}}(\mathbb{P}\times\mathbb{P}) =$ Zero section of $T^*(\mathbb{P}\times\mathbb{P})$.

$\Omega^1_{\mathbb{P}\times\mathbb{P}/\mathbb{P}}$ = relative 1-forms along $pr_1 : \mathbb{P}\times\mathbb{P} \to \mathbb{P}$.

$Q := \pi_Y^* \Omega^1_{\mathbb{P}\times\mathbb{P}/\mathbb{P}}$, $\pi_Y : Z_Y \xrightarrow{\sim} \overline{Y}$.

$\forall n \in \mathbb{Z}$, $\mathcal{O}_n := \pi_\Delta^* \mathcal{O}_{\mathbb{P}}(n)$, $\pi_\Delta : Z_\Delta \to \mathbb{P}_\Delta$.

Affine Hecke alg $\mathbb{H} \ni T, X, X^{-1}$. $X = e^{\varpi}$, ϖ = fundamental weight.

$(T+1)(T-q) = 0$, $X \cdot X^{-1} = X^{-1} \cdot X = 1$,

$T X^{-1} - X T = (1-q) X$.

Let $c = -(T+1)$

Define: $\Theta : \{c, X, X^{-1}\} \to K^{G\times\mathbb{C}^*}(Z)$

$c \mapsto [qQ]$, $X \mapsto [\mathcal{O}_{-1}]$, $X^{-1} \mapsto [\mathcal{O}_1]$

Thm: Θ extends to an alg. homomorphism. $\Theta: \mathbb{H} \to K^{G\times\mathbb{C}^*}(Z)$.

i.e. $(qQ)*(qQ) = -(q+1)qQ$,

$(qQ)*\mathcal{O}_1 - \mathcal{O}_1*(qQ) = q\mathcal{O}_{-1} - \mathcal{O}_0$.

$\mathcal{O}_1 * \mathcal{O}_{-1} = \mathcal{O}_0$

pf: The last relation is obvious. Let's check the first two.

$\overline{Y} \simeq \mathbb{P}\times\mathbb{P} \xrightarrow{pr_1} \mathbb{P}$

$Q = \mathcal{O}_{\mathbb{P}} \boxtimes \Omega^1_{\mathbb{P}}$

$T^*\mathbb{P}\times T^*\mathbb{P} \xrightarrow{\pi\times id} \mathbb{P}\times T^*\mathbb{P}$

$\rightsquigarrow Z \xhookrightarrow{\overline{\pi}} \mathbb{P}\times T^*\mathbb{P} \xhookleftarrow{\overline{i}} \mathbb{P}\times\mathbb{P}$ $\qquad \overline{i} = id \times (\text{zero section})$

To perform convolution, need to know the class Q in $\mathbb{P}\times T^*\mathbb{P}$.

$\mathbb{P} \underset{\pi}{\overset{i}{\rightleftarrows}} T^*\mathbb{P}$ $\qquad$ i.e. $\mathcal{O}_{\mathbb{P}} \boxtimes i_*\Omega^1_{\mathbb{P}}$.

Koszul resolution $\rightsquigarrow$

$$0 \to q^{-1}\pi^*T\mathbb{P} \to \mathcal{O}_{T^*\mathbb{P}} \to i_*\mathcal{O}_{\mathbb{P}} \to 0.$$

Rmk. 1) To restore the $\mathbb{C}^*$-equiv. we need to times $\pi^*T\mathbb{P}$ by q^{-1}

recall $\mathbb{C}^*$ acts on fibers of $T^*\mathbb{P}$ by q according to our convention.

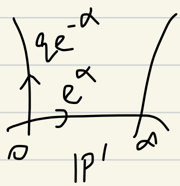
Hence, $\mathbb{C}^*$ acts on fiber of $T\mathbb{P}$ by q^{-1}.

2) This correction factor is the same as in [CG, P393] However, in [CG], $q \rightsquigarrow (z \mapsto z)$. But here, $q \rightsquigarrow (z \mapsto z^{-1})$

I believe [CG] has a mistake about this.

3) another evidence. (More evidence later)

$$[i_* \mathcal{O}_{\mathbb{P}}] = [\mathcal{O}_{T^*\mathbb{P}}] - q^{-1}[\pi^* T\mathbb{P}].$$



$(\mathbb{P})^T = \{0, \alpha\} = W$, $\quad 0 \Leftrightarrow id$

$T_0\mathbb{P} = e^{\alpha}$ $\quad \alpha$ = positive root = T-weight in $\mathfrak{g}/\mathfrak{b}$.

$T_0^*\mathbb{P} = qe^{-\alpha}$ (another reason to add this $\mathbb{C}^*$-char. explicitly)

Hence, $[i_* \mathcal{O}_{\mathbb{P}}]|_0 = \lambda\left((T_{\mathbb{P}} T^*\mathbb{P}^1)^\vee|_0\right)$

$T_{\mathbb{P}}(T^*\mathbb{P}^1)$ = normal bundle.

$T_{\mathbb{P}}(T^*\mathbb{P})|_0$ has $T\times\mathbb{C}^*$-weight $qe^{-\alpha}$.

Thus, $[i_* \mathcal{O}_{\mathbb{P}}]|_0 = 1 - q^{-1}e^{\alpha}$

$$= [\mathcal{O}_{T^*\mathbb{P}}]|_0 - q^{-1}[\pi^* T\mathbb{P}]|_0$$

Koszul resolution

$$0 \to q^{-1}\pi^* T\mathbb{P} \to \mathcal{O}_{T^*\mathbb{P}} \to i_* \mathcal{O}_{\mathbb{P}} \to 0.$$

Tensor with $\pi^*\Omega^1_{\mathbb{P}}$, we get

$$0 \to q^{-1}\mathcal{O}_{T^*\mathbb{P}} \to \pi^*\Omega^1_{\mathbb{P}} \to i_*\Omega^1_{\mathbb{P}} = i_*\mathcal{O}_{\mathbb{P}} \otimes \pi^*\Omega^1_{\mathbb{P}} \to 0$$

$$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \, i_*(\mathcal{O}_{\mathbb{P}} \otimes i^*\pi^*\Omega^1_{\mathbb{P}}).$$

$$\rightsquigarrow 0 \to q^{-1}\mathcal{O}_{\mathbb{P}} \boxtimes \mathcal{O}_{T^*\mathbb{P}} \to \mathcal{O}_{\mathbb{P}} \boxtimes \pi^*\Omega^1_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}} \boxtimes i_*\Omega^1_{\mathbb{P}} \to 0.$$

$\rightsquigarrow$ in $K^{G\times\mathbb{C}^*}(\mathbb{P}\times T^*\mathbb{P})$,

$$qQ = q\,\mathcal{O}_{\mathbb{P}} \boxtimes i_*\Omega^1_{\mathbb{P}}$$
$$= q\,\mathcal{O}_{\mathbb{P}} \boxtimes \pi^*\Omega^1_{\mathbb{P}} - \mathcal{O}_{\mathbb{P}} \boxtimes \mathcal{O}_{T^*\mathbb{P}}.$$

$K^{G\times\mathbb{C}^*}(T^*\mathbb{P}\times\mathbb{P})$ $\in$

Hence, $(qQ) * (qQ)$

$$= \left(q\,\mathcal{O}_{\mathbb{P}} \boxtimes \pi^*\Omega^1_{\mathbb{P}} - \mathcal{O}_{\mathbb{P}} \boxtimes \mathcal{O}_{T^*\mathbb{P}}\right) * \left(q\, i_*\mathcal{O}_{\mathbb{P}} \boxtimes \Omega^1_{\mathbb{P}}\right)$$

$$= q^2 \langle \pi^*\Omega^1_{\mathbb{P}}, i_*\mathcal{O}_{\mathbb{P}} \rangle \cdot \mathcal{O}_{\mathbb{P}} \boxtimes \Omega^1_{\mathbb{P}} - q\langle \mathcal{O}_{T^*\mathbb{P}}, i_*\mathcal{O}_{\mathbb{P}} \rangle \cdot \mathcal{O}_{\mathbb{P}} \boxtimes \Omega^1_{\mathbb{P}}$$

$$= q^2 Q \cdot p_*\pi_*(\pi^*\Omega^1_{\mathbb{P}} \otimes i_*\mathcal{O}_{\mathbb{P}}) - qQ \cdot p_*\pi_*(i_*\mathcal{O}_{\mathbb{P}})$$

$$= q^2 Q \cdot p_*\mathcal{O}(-2) - qQ\, p_*(\mathcal{O}_{\mathbb{P}})$$

$$= -q^2 Q - qQ = -(q+1)qQ.$$

$$\begin{array}{c} T^*\mathbb{P} \\ i\uparrow\downarrow\pi \\ \mathbb{P} \\ \downarrow p \\ pt \end{array}$$

Now Let's prove the second relation.

Recall $Z \xhookrightarrow{\bar{\pi}} \mathbb{P} \times T^*\mathbb{P} \xhookleftarrow{\bar{i}} \mathbb{P} \times \mathbb{P}$.

we need two facts.

1) $\Phi := \bar{i}^* \bar{\pi}_* : K^{G\times\mathbb{C}^*}(Z) \to K^{G\times\mathbb{C}^*}(\mathbb{P}\times\mathbb{P})$

is an alg homormorphism. See Cor 5.4.14 in [CG]

2) Φ is injective. (will be proved later for $\mathcal{B}$)

Thus, we only need to verify

$$\Phi(qQ) * \Phi(\mathcal{O}_1) - \Phi(\mathcal{O}_{-1}) * \Phi(qQ) = q\,\Phi(\mathcal{O}_{-1}) - \Phi(\mathcal{O}_1) \qquad (*)$$

in $K^{G\times\mathbb{C}^*}(\mathbb{P}\times\mathbb{P}) \underset{\text{Künneth}}{\simeq} K^{G\times\mathbb{C}^*}(\mathbb{P}) \otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(\mathbb{P})$

Write $\mathcal{O}_\Delta(u) := \Delta_* \mathcal{O}(u)$, $\Delta : \mathbb{P} \hookrightarrow \mathbb{P}\times\mathbb{P}$.

$qQ = q\,\mathcal{O}_{\mathbb{P}^1} \boxtimes \pi^*\Omega'_{\mathbb{P}} - \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{T^*\mathbb{P}}$

$\rightsquigarrow \Phi(qQ) = q\,\mathcal{O} \boxtimes \mathcal{O}(-2) - \mathcal{O} \boxtimes \mathcal{O}$.

$\Phi(\mathcal{O}_n) = \mathcal{O}_\Delta(u)$

Recall the general fact, $\forall\, \mathcal{L} \in K^{G\times\mathbb{C}^*}(\mathbb{P})$, $\mathcal{F} \in K^{G\times\mathbb{C}^*}(\mathbb{P}\times\mathbb{P})$, $\Delta: \mathbb{P} \hookrightarrow \mathbb{P}\times\mathbb{P}$

$$\mathcal{F} * \Delta_* \mathcal{L} = \mathcal{F} \otimes pr_2^* \mathcal{L} \qquad \Delta_* \mathcal{L} * \mathcal{F} = pr_1^* \mathcal{L} \otimes \mathcal{F}.$$

$$\begin{tikzcd}
\mathbb{P}\times\mathbb{P} \arrow[r, hook, "id\times\Delta"] \arrow[d, "pr_2"'] \arrow[dr, phantom, "\square"] & \mathbb{P}\times\mathbb{P}\times\mathbb{P} \arrow[d, "p_{23}"] \arrow[l, bend right, "p_{12}"'] \\
\mathbb{P} \arrow[r, hook, "\Delta"] & \mathbb{P}\times\mathbb{P}
\end{tikzcd}$$

$$\mathbb{P}\times\mathbb{P}\times\mathbb{P} \xrightarrow{p_{ij}} \mathbb{P}\times\mathbb{P}$$

$$\begin{aligned}
\mathcal{F} * \Delta_* \mathcal{L} &= p_{13*}(p_{12}^* \mathcal{F} \otimes p_{23}^* \Delta_* \mathcal{L}) \\
&= p_{13*}(p_{12}^* \mathcal{F} \otimes (id\times\Delta)_* pr_2^* \mathcal{L}) \\
&= p_{13*}(id\times\Delta)_*((id\times\Delta)^* p_{12}^* \mathcal{F} \otimes pr_2^* \mathcal{L}) \\
&= \mathcal{F} \otimes pr_2^* \mathcal{L}.
\end{aligned}$$

Thus LHS of $(*)$

$$= q\,\mathcal{O}\boxtimes\mathcal{O}(-1) - \mathcal{O}\boxtimes\mathcal{O}(1) - q\,\mathcal{O}(-1)\boxtimes\mathcal{O}(-2) + \mathcal{O}(-1)\boxtimes\mathcal{O}.$$

For the RHS, we need Beilinson's resolution.

$$0 \to \mathcal{O}(-1)\boxtimes\Omega^1(1) \to \mathcal{O}\boxtimes\mathcal{O} \to \mathcal{O}_\Delta \to 0$$

$$\rightsquigarrow\ 0 \to \mathcal{O}(-1)\boxtimes\mathcal{O}(-2) \to \mathcal{O}\boxtimes\mathcal{O}(-1) \to \mathcal{O}_\Delta(-1) \to 0$$

and $0 \to \mathcal{O}(-1)\boxtimes\mathcal{O} \to \mathcal{O}\boxtimes\mathcal{O}(1) \to \mathcal{O}_\Delta(1) \to 0.$

Thus, RHS of $(*) = q\,\mathcal{O}\boxtimes\mathcal{O}(-1) - q\,\mathcal{O}(-1)\boxtimes\mathcal{O}(-2) - \mathcal{O}\boxtimes\mathcal{O}(1) + \mathcal{O}(-1)\boxtimes\mathcal{O}.$ $\square$

$\Rightarrow$ $(*)$ holds

Rmks. Compare with Lusztig's approach.

Recall Lusztig:

$\forall [\mathcal{F}] \in K^G(G/B)$,

$$T_s([\mathcal{F}]) = \pi_s^* \pi_{s*}[\mathcal{F}] - [\mathcal{F}] - q\pi_s^*\pi_{s*}([\mathcal{F}] \otimes [\Omega_s^1])$$

$\pi_s: G/B \to G/P_s$

$\pi: \mathbb{P} \to pt$

$\forall \lambda \in X^*(T)$,

$$e^\lambda([\mathcal{F}]) := [\mathcal{F}] \otimes [\mathcal{L}_\lambda^*]$$

Thus, $c = -(T_s+1) \mapsto q\,\mathcal{O} \boxtimes \Omega_{\mathbb{P}}^1 - \mathcal{O} \boxtimes \mathcal{O}$

$$= \Phi(qQ)$$

$$e^\varpi \mapsto \mathcal{L}_\varpi^* \otimes - = \mathcal{O}(-1) \otimes - = \text{convolution with } \mathcal{O}_\Delta(-1)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Phi(\mathcal{O}_{-1})$$

Thus, Lusztig's construction

= the one above.

(this is another reason for the normalization of $q \rightsquigarrow (z \mapsto z^{-1})$
st $\mathbb{C}^* \curvearrowright T^*\mathbb{P}$ by char. q).

Thm. The alg homomorphism

$$\Theta : \mathbb{H} \to K^{G\times\mathbb{C}^*}(Z) \text{ is an isomorphism.}$$

pf. give a filtration of $\mathbb{H}$ and $K^{G\times\mathbb{C}^*}(Z)$ as follows:

$$\begin{array}{ccc} 0 \subseteq \mathbb{H}_0 \subseteq \mathbb{H} & & 0 \subseteq K^{G\times\mathbb{C}^*}(Z_\Delta) \subseteq K^{G\times\mathbb{C}^*}(Z) \\ \| & & \text{\rotatebox{90}{$\cong$}} \\ \text{subalg gen by } X \text{ and } X^{-1} & & K^{G\times\mathbb{C}^*}(\mathbb{P}) \\ \text{\rotatebox{90}{$\cong$}} & & \\ R(T)[q,q^{-1}] & \xrightarrow{\sim} & \end{array}$$

Thus, Θ preserves the filtration.

Θ is an isomorphism $\Longleftrightarrow$ $\mathrm{gr}\,\Theta$ is an isomorphism

We already showed $\Theta : \mathbb{H}_0 \xrightarrow{\sim} K^{G\times\mathbb{C}^*}(Z_\Delta)$.

$$Z_\Delta \xrightarrow{i} Z \overset{j}{\hookleftarrow} T^*_Y(\mathbb{P}\times\mathbb{P}), \qquad Y = \mathbb{P}\times\mathbb{P}\setminus\Delta.$$

$$\begin{tikzcd} \rightsquigarrow K_1^{G\times\mathbb{C}^*}(Z) \arrow[r, "j^*"] & K_1^{G\times\mathbb{C}^*}(T^*_Y(\mathbb{P}\times\mathbb{P})) \arrow[r, "\partial"] & K^{G\times\mathbb{C}^*}(Z_\Delta) \arrow[r] & K^{G\times\mathbb{C}^*}(Z) \arrow[r] & K^{G\times\mathbb{C}^*}(T^*_Y(\mathbb{P}\times\mathbb{P})) \arrow[r] & 0 \\ K_1^{G\times\mathbb{C}^*}(\mathbb{B}) \arrow[u, "\pi^*"] \arrow[ur, "\pi^*"', "\sim"] \end{tikzcd}$$

Thom iso

$\Rightarrow \mathrm{Coker}\, j^* = 0 \Rightarrow \mathrm{Ker}\,\partial = 0.$

$Z \hookrightarrow T^*\mathbb{P}\times T^*\mathbb{P} \xrightarrow{pr_2} T^*\mathbb{P} \to \mathbb{P}$

$$\begin{tikzcd} Z \arrow[r, hookleftarrow, "j"] \arrow[dr, "\pi"'] & T^*_Y(\mathbb{P}\times\mathbb{P}) \arrow[d, "\pi"] \\ & \mathbb{P} \end{tikzcd}$$

is an affine fibration with fiber $\mathbb{P}\setminus\{pt\}$

Thus,

$$0 \to K^{G\times\mathbb{C}^*}(Z_\Delta) \to K^{G\times\mathbb{C}^*}(Z) \to K^{G\times\mathbb{C}^*}(T_Y^*(\mathbb{P}\times\mathbb{P})) \to 0.$$

Moreover, $K^{G\times\mathbb{C}^*}(T_Y^*(\mathbb{P}\times\mathbb{P})) \underset{\text{Thom.}}{\simeq} K^{G\times\mathbb{C}^*}(\mathbb{P}) \simeq R(T\times\mathbb{C}^*)$

$$\mathrm{gr}\,\Theta : \mathbb{H}/\mathbb{H}_0 \to K^{G\times\mathbb{C}^*}(Z)/K^{G\times\mathbb{C}^*}(Z_\Delta) \simeq K^{G\times\mathbb{C}^*}(T_Y^*(\mathbb{P}\times\mathbb{P}))$$

sends $T \mapsto u[\mathcal{O}_{T_Y^*(\mathcal{B}\times\mathcal{B})}]$, $u \in R(T\times\mathbb{C}^*)$ invertible.

$\leadsto$ $\mathrm{gr}\,\Theta : \mathbb{H}_0 \oplus \mathbb{H}/\mathbb{H}_0 \to K^{G\times\mathbb{C}^*}(Z_\Delta) \oplus K^{G\times\mathbb{C}^*}(Z)/K^{G\times\mathbb{C}^*}(Z_\Delta)$

is an isomorphism

$\leadsto$ Θ is an isomorphism. $\square$

§. Proof of the Main Theorem. $K^{G\times\mathbb{C}^*}(Z) \simeq \mathbb{H}$.

generators of $\mathbb{H}$,

$$S = \{e^\lambda \mid \lambda \in P\} \cup \{T_s \mid s \text{ simple reflection in } W\}$$

We first construct a map $\Theta: S \to K^{G\times\mathbb{C}^*}(Z)$

$\mathcal{B}_\Delta \subseteq \mathcal{B}\times\mathcal{B}$ diagonal,

$Z_\Delta := T^*_{\mathcal{B}_\Delta}(\mathcal{B}\times\mathcal{B}) \xrightarrow{\pi_\Delta} \mathcal{B}_\Delta$

$\forall \lambda \in P$. $\mathcal{L}_\lambda = G\times_B \mathbb{C}_\lambda \in K^{G\times\mathbb{C}^*}(\mathcal{B}_\Delta)$

$\mathcal{O}_\lambda := \pi_\Delta^* \mathcal{L}_\lambda$,

$$\Theta(e^\lambda) = [\mathcal{O}_{-\lambda}].$$

$\forall$ simple reflection $s \in W$. $Y_s \subseteq \mathcal{B}\times\mathcal{B}$ the corr. G-orbit.

$$\begin{tikzcd} \overline{Y_s} = Y_s \sqcup \mathcal{B}_\Delta \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\square"] & \mathcal{B} \arrow[d] \\ \mathcal{B} \arrow[r] & G/P_s \end{tikzcd}$$

$\Omega^1_{\overline{Y_s}/\mathcal{B}}$ = sheaf of relative 1-forms w.r.t. pr_1

$\pi_s: T^*_{\overline{Y_s}}(\mathcal{B}\times\mathcal{B}) \to \overline{Y_s}$

Let $\mathcal{Q}_s = \pi_s^* \Omega^1_{\overline{Y_s}/\mathcal{B}}$

define $\Theta(T_s) = -[q\mathcal{O}_s] - [\mathcal{O}_0]$,

Then the first goal is to show that Θ extends to an alg homomorphism $\mathbb{H} \to K^{G\times\mathbb{C}^*}(Z)$

<u>idea:</u> Construct a $\mathbb{H}$ and $K^{G\times\mathbb{C}^*}(Z)$ module M.

$\rho_1\ \mathbb{H} \to \mathrm{End}\,M$, $\rho_2 : K^{G\times\mathbb{C}^*}(Z) \to \mathrm{End}\,M$.

We show 1) $\rho_1(w) = \rho_2(\Theta(w))$

2) ρ_2 is injective.

$\Rightarrow \Theta : \mathbb{H} \to K^{G\times\mathbb{C}^*}(Z)$ is an alg. homomorphism.

Answer $M = K^{G\times\mathbb{C}^*}(T^*\mathcal{B}) \simeq R(T)[q,q^{-1}]$.

Let's first work on ρ_1

finite Hecke alg.

Let $e. = \sum_w T_w \in H_W \subseteq \mathbb{H}$.

<u>Lemma</u>: 1) $T_w \mapsto q^{\ell(w)}$ extends to an alg homomorphism

$\varepsilon . H_W \to \mathbb{Z}[q,q^{-1}]$

2) $\forall a \in H_W$, $a.e = e a = \varepsilon(a) e$.

3) $\mathbb{H}\cdot e$ is a free $R(T)[q,q^{-1}]$-mod with generator e,

and $\mathbb{H}e \simeq \mathrm{Ind}_{H_W}^{\mathbb{H}} \varepsilon$.

$\rightsquigarrow \rho_{\mathbb{H}}: \mathbb{H} \to \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}(\mathbb{H}\cdot e)$

The $K^{G\times\mathbb{C}^*}(Z)$-action.

$$K^{G\times\mathbb{C}^*}(T^*\mathcal{B}) \underset{\text{Thom}}{\simeq} K^{G\times\mathbb{C}^*}(\mathcal{B}) \simeq R(T)[q,q^{-1}] \xrightarrow{\sim} \mathbb{H}\cdot e$$
$$e^{\lambda} \longmapsto e^{-\lambda} e.$$

By convolution,

$$\rho_{T^*\mathcal{B}}: K^{G\times\mathbb{C}^*}(Z) \longrightarrow \mathrm{End}_{R(G\times\mathbb{C}^*)}(K^{G\times\mathbb{C}^*}(T^*\mathcal{B})).$$

---

claim I: $\rho_{T^*\mathcal{B}}$ is injective.

claim II:

$$\begin{tikzcd}
S \arrow[r] & \mathbb{H} \arrow[r, "\rho_{\mathbb{H}}"] \arrow[dl, "\Theta"', "\wr"] \arrow[d, phantom, "\circlearrowright"] & \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}(\mathbb{H}\cdot e) \arrow[d, "\Phi", "\wr"'] \\
K^{G\times\mathbb{C}^*}(Z) \arrow[rr, "\rho_{T^*\mathcal{B}}"] & & \mathrm{End}_{\mathbb{Z}[q,q^{-1}]} K^{G\times\mathbb{C}^*}(T^*\mathcal{B})
\end{tikzcd}$$

Let's assume these two claims first.

<u>Prop</u>. Θ can be extended to an alg homomorphism $\mathbb{H} \to K^{G\times\mathbb{C}^*}(Z)$.

pf. Let $T(S)$ = free alg generated by S over $\mathbb{Z}[q,q^{-1}]$.

$$\begin{tikzcd} T(S) \arrow[r, "I"] \arrow[dr, "\hat{\Theta}"'] & \mathbb{H} \arrow[d, dashed, "\exists\,\Theta"] \\ & K^{G\times\mathbb{C}^*}(Z) \end{tikzcd}$$

We only need to show for any $a \in T(S)$, s.t. $I(a)=0$, then $\hat{\Theta}(a)=0$.

Since $\rho_{T^*\mathcal{B}} \circ \hat{\Theta}(a) = \Phi \circ \rho_{\mathbb{H}} \circ I(a) = 0$

claim I $\Rightarrow$ $\hat{\Theta}(a)=0$. $\square$

<u>Thm</u>. The alg. homomorphism $\Theta: \mathbb{H} \to K^{G\times\mathbb{C}^*}(Z)$ in the previous prop is an isomorphism.

Sketch: We introduce filtrations on both $\mathbb{H}$ and $K^{G\times\mathbb{C}^*}(Z)$, st

Θ is filtration preserving, and $\mathrm{gr}\,\Theta$ is an isomorphism

$\Rightarrow$ Θ is an isomorphism filtration on $K^{G\times\mathbb{C}^*}(Z)$:

$$Y_w = G(B, wB) \subseteq \mathcal{B}\times\mathcal{B}, \qquad Z_{\le w} := \bigsqcup_{y\le w} T^*_{Y_y}(\mathcal{B}\times\mathcal{B})$$

Then $K^{G\times\mathbb{C}^*}(Z_{\le w}) \hookrightarrow K^{G\times\mathbb{C}^*}(Z)$ gives a filtration of $K^{G\times\mathbb{C}^*}(Z)$,

and $K^{G\times\mathbb{C}^*}(Z_{\le w}) / K^{G\times\mathbb{C}^*}(Z_{<w}) \simeq K^{G\times\mathbb{C}^*}(T^*_{Y_w}(\mathcal{B}\times\mathcal{B}))$, a

free $R(G\times\mathbb{C}^*)$-mod. with generator $[\mathcal{O}_{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})}]$

---

filtration on $\mathbb{H}$.

$$\mathbb{H}_{\le w} := \mathrm{Span}\{e^\lambda T_y \mid \lambda \in P, y \le w\}$$

Then we have

Prop. (7.6.12)

1) $\Theta(\mathbb{H}_{\le w}) \subseteq K^{G\times\mathbb{C}^*}(Z_{\le w})$

2) $\Theta : \mathbb{H}_{\le w}/\mathbb{H}_{<w} \longrightarrow K^{G\times\mathbb{C}^*}(Z_{\le w}) / K^{G\times\mathbb{C}^*}(Z_{<w}) \simeq K^{G\times\mathbb{C}^*}(T^*_{Y_w}(\mathcal{B}\times\mathcal{B}))$

$$T_w \mapsto c_w \cdot [\mathcal{O}_{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})}], \quad c_w \in R(T\times\mathbb{C}^*) \text{ is invertible.}$$

(pf uses the fact $\overline{Y_{s_1}} \times_{\mathcal{B}} \overline{Y_{s_2}} \times \cdots \times \overline{Y_{s_l}} \twoheadrightarrow \overline{Y_w}$ $\quad w = s_1 \cdots s_l$ reduced, and $Y_{s_1}\times_{\mathcal{B}} Y_{s_2}\times\cdots\times Y_{s_l}\xrightarrow{\sim} Y_w$)

Now let's prove claim I and II.

$$T^*\mathcal{B}\times T^*\mathcal{B}\xrightarrow{id\times\pi}\mathcal{B}\times T^*\mathcal{B}$$

$$\begin{tikzcd}
\cup & & \\
Z \arrow[r, hook, "\bar{j}"] & \mathcal{B}\times T^*\mathcal{B} & \mathcal{B}\times\mathcal{B} \arrow[l, hook', "\bar{i}"']
\end{tikzcd}$$

<u>Introduce</u>

$$\begin{tikzcd}
 & \mathbb{H} \arrow[r, "\rho_{\mathbb{H}}"] & \mathrm{End}(\mathbb{H}\cdot e) \arrow[rr, equal, "\beta"] & & \mathrm{End}\,R(T)[q,q^{-1}] \\
S \arrow[ur] \arrow[r, "\theta"] & K^{G\times\mathbb{C}^*}(Z) \arrow[r, "\rho_{T^*\mathcal{B}}"] \arrow[d, "\bar{i}^*\bar{j}_*"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{End}\,K^{G\times\mathbb{C}^*}(T^*\mathcal{B}) \arrow[d, equal, "\mathrm{Th}"] \arrow[rr, equal, "\mathrm{Thod}"] & & \\
 & K^{G\times\mathbb{C}^*}(\mathcal{B}\times\mathcal{B}) \arrow[r, "\rho_{\mathcal{B}}"] & \mathrm{End}\,K^{G\times\mathbb{C}^*}(\mathcal{B}) \arrow[uurr, equal, "\alpha"'] & &
\end{tikzcd}$$

$\beta(e^{\lambda})=e^{-\lambda}$

Cor 5.4.34

Künneth theorem $\Rightarrow$ $\rho_{\mathcal{B}}$ is an isomorphism.

thus Claim I: $\rho_{T^*\mathcal{B}}$ is injective

$\Leftarrow$ $\bar{i}^*\bar{j}_*$ is injective (we used this in the $SL(2)$ example)

The proof reduces to $T\times\mathbb{C}^*$-equiv. K-theory, and uses localization.

$$Z \overset{\bar{j}}{\hookrightarrow} \mathcal{B}\times T^*\mathcal{B} \xrightarrow{pr_1} \mathcal{B}.$$

Z_b := fiber over $b = \bigsqcup_{w\in W} T^*_{\mathcal{B}_w}\mathcal{B}$

$$\begin{tikzcd}
Z \arrow[r, hook, "\bar{j}"] & \mathcal{B}\times T^*\mathcal{B} & \mathcal{B}\times\mathcal{B} \arrow[l, hook', "\bar{i}"'] \arrow[r, "pr_1"] & \mathcal{B} \\
\{b\}\times Z_b \arrow[u, hook] \arrow[r, hook, "\bar{j}"] & \{b\}\times T^*\mathcal{B} \arrow[u, hook] & \{b\}\times\mathcal{B} \arrow[l, hook', "i"'] \arrow[u, hook] \arrow[r] & \{b\} \arrow[u, hook]
\end{tikzcd}$$

$X = G\times_B Y$

$Y \hookrightarrow X$

$\rightsquigarrow K^{G\times\mathbb{C}^*}(X) \simeq K^{B\times\mathbb{C}^*}(Y)$
$= K^{T\times\mathbb{C}^*}(Y)$

$$\rightsquigarrow \begin{tikzcd}
K^{G\times\mathbb{C}^*}(Z) \arrow[r, "\bar{i}^*\circ\bar{j}_*"] \arrow[d, equal, "res"'] \arrow[dr, phantom, "\circlearrowleft"] & K^{G\times\mathbb{C}^*}(\mathcal{B}\times\mathcal{B}). \arrow[d, equal, "res"] \\
K^{T\times\mathbb{C}^*}(Z_b) \arrow[r, "i^*\circ j_*"] & K^{T\times\mathbb{C}^*}(\mathcal{B})
\end{tikzcd}$$

pf of claim I. $\rho_{T^*\mathcal{B}}$ inj $\Leftarrow$ $\bar{i}^*\circ\bar{j}_*$ inj $\Leftarrow$ $i^*\circ j_*$ is inj

cellular fibration of Z_b, $T^*\mathcal{B}$, and $\mathcal{B}$.

$\rightsquigarrow K^{T\times\mathbb{C}^*}(Z_b)$, $K^{T\times\mathbb{C}^*}(T^*\mathcal{B})$, and $K^{T\times\mathbb{C}^*}(\mathcal{B})$ are freely $K^{T\times\mathbb{C}^*}$-modules

Thus, we can check injectivity after localization.

On the other hand, $(Z_b)^{T\times\mathbb{C}^*} = (T^*\mathcal{B})^{T\times\mathcal{C}^*} = (\mathcal{B})^{T\times\mathbb{C}^*} \overset{1:1}{\leftrightarrow} W$.

$\Rightarrow$ injectivity after localization. $\square$

Now let's check claim II.

<u>Recall</u> $\quad Z \xhookrightarrow{\bar{j}} T^*\mathcal{B}\times\mathcal{B} \xhookleftarrow{\bar{i}} \mathcal{B}\times\mathcal{B}$

$$\begin{tikzcd}
 & \mathbb{H} \arrow[r, "\rho_{\mathbb{H}}"] & \mathrm{End}(\mathbb{H}\cdot e) \arrow[rr, equal, "\beta"] & & \mathrm{End}\,R(T)[q,q^{-1}] \\
S \arrow[ur, "\mathrm{incl}"] \arrow[r, "\Theta"] & K^{G\times\mathbb{C}^*}(Z) \arrow[r, "\rho_{T^*\mathcal{B}}"] \arrow[d, "\bar{i}^*\bar{j}_*"'] \arrow[dr, phantom, "\circlearrowleft\ \text{Cor 5.4.34}"] & \mathrm{End}\,K^{G\times\mathbb{C}^*}(T^*\mathcal{B}) \arrow[rr, equal, "\alpha\circ Th"] \arrow[d, equal, "Th"] & & \mathrm{End}\,R(T)[q,q^{-1}] \\
 & K^{G\times\mathbb{C}^*}(\mathcal{B}\times\mathcal{B}) \arrow[r, "\rho_{\mathcal{B}}"] & \mathrm{End}\,K^{G\times\mathbb{C}^*}(\mathcal{B}) \arrow[urr, equal, "\alpha"'] & &
\end{tikzcd}$$

$\beta(e^{\lambda}) = e^{-\lambda}$

Let $\psi_1 = \beta\circ\rho_{\mathbb{H}}\circ\mathrm{incl}$, $\psi_2 = \alpha\circ Th\circ\rho_{T^*\mathcal{B}}\circ\Theta$

$$\psi_2 = \psi_3 = \alpha\circ\rho_{\mathcal{B}}\circ\bar{i}^*\bar{j}_*\circ\Theta$$

Now we focus on step 2.

Recall

$$\begin{tikzcd} Z \arrow[r, hook, "\bar{j}"] & \mathcal{B}\times T^*\mathcal{B} & \mathcal{B}\times\mathcal{B} \arrow[l, hook', "\bar{i}"'] \arrow[r, "pr_1"] & \mathcal{B} \\ \{b\}\times Z_b \arrow[u, hook] \arrow[r, hook, "j"] & \{b\}\times T^*\mathcal{B} \arrow[u, hook] & \{b\}\times\mathcal{B} \arrow[l, hook', "i"'] \arrow[u, hook] \arrow[r] & \{b\} \arrow[u, hook] \end{tikzcd}$$

$X = G\times_B Y$

$Y \hookrightarrow X$

$\rightsquigarrow K^{G\times\mathbb{C}^*}(X) \simeq K^{B\times\mathbb{C}^*}(Y)$
$= K^{T\times\mathbb{C}^*}(Y)$

$$\rightsquigarrow \begin{tikzcd} K^{G\times\mathbb{C}^*}(Z) \arrow[r, "\bar{i}^*\circ\bar{j}_*"] \arrow[d, "res", equal] \arrow[dr, phantom, "\circlearrowleft"] & K^{G\times\mathbb{C}^*}(\mathcal{B}\times\mathcal{B}). \arrow[d, "res", equal] \\ K^{T\times\mathbb{C}^*}(Z_b) \arrow[r, "i^*\circ j_*"] & K^{T\times\mathbb{C}^*}(\mathcal{B}) \end{tikzcd}$$

$\overline{Y_s} = Y_s \sqcup \mathcal{B}_\Delta = \mathcal{B}\times_{G/P_s}\mathcal{B}$
$\quad\downarrow pr_1$
$\quad\mathcal{B}$

$Q_s := \pi_s^*\Omega^1_{\overline{Y_s}/\mathcal{B}}$

$\pi_s : T^*_{\overline{Y_s}}(\mathcal{B}\times\mathcal{B}) \to \overline{Y_s}$

$\mathcal{O}(-T_s-1) = [q\, Q_s]$

<u>Lemma</u>: $\bar{i}^*\circ\bar{j}_*(q\, Q_s) = q\cdot\Omega^1_{\overline{Y_s}/\mathcal{B},\, pr_1} - \mathcal{O}_{\overline{Y_s}} \in K^{G\times\mathbb{C}^*}(\mathcal{B}\times\mathcal{B})$.

pf. Let's first prove $res\circ\bar{i}^*\circ\bar{j}_*[q\, Q_s] = \varepsilon_*(q\cdot[\Omega^1_{\overline{\mathcal{B}_s}}] - [\mathcal{O}_{\overline{\mathcal{B}_s}}])$,
where $\varepsilon : \overline{\mathcal{B}_s} \hookrightarrow \mathcal{B}$.

first of all, $\mathrm{res} \circ \bar{i}^* \circ \bar{j}_* = i^* \circ j_* \circ \mathrm{res}$.

$$\begin{tikzcd}
\{b\} \times T^*_{\bar{\mathcal{B}}_s}\mathcal{B} \arrow[r, hook] \arrow[d, "\pi_s"'] \arrow[dr, phantom, "\square"] & T^*_{\bar{Y}_s}(\mathcal{B}\times\mathcal{B}) \arrow[d, "\pi_s"] \\
\{b\} \times \bar{\mathcal{B}}_s \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \bar{Y}_s \arrow[d, "pr_1"] \\
\{b\} \arrow[r, hook] & \mathcal{B}
\end{tikzcd}$$

$\rightsquigarrow \mathrm{res}(\Omega^1_{\bar{Y}_s/\mathcal{B}}) = \Omega^1_{\bar{\mathcal{B}}_s}$

$\rightsquigarrow \mathrm{res}(\mathcal{B}_s) = \pi_s^* \Omega^1_{\bar{\mathcal{B}}_s}$.

Thus, we need to compute $i^* \circ j_* \pi_s^* \Omega^1_{\bar{\mathcal{B}}_s}$.

where $Z_b = \bigsqcup_w T^*_{\bar{\mathcal{B}}_s}\mathcal{B} \xrightarrow{j} T^*\mathcal{B} \xleftarrow{i} \mathcal{B}$.

We have

$$\begin{tikzcd}
 & T^*\mathcal{B}|_{\bar{\mathcal{B}}_s} \arrow[r, hook, "\bar{\varepsilon}"] & T^*\mathcal{B} \\
T^*_{\bar{\mathcal{B}}_s}\mathcal{B} \arrow[ur, "\tilde{j}"] \arrow[urr, bend left, "j"] \arrow[dr, bend right, "\pi_s"'] & & \\
 & \bar{\mathcal{B}}_s \arrow[uu, "\bar{i}"] \arrow[ul, "i_s"] \arrow[r, hook, "\varepsilon"] \arrow[uur, phantom, "\square"] & \mathcal{B} \arrow[uu, "i"']
\end{tikzcd}$$

$\Rightarrow$

$$i^* \circ j_* \circ \pi_s^* \Omega^1_{\bar{\mathcal{B}}_s} \qquad j = \bar{\varepsilon} \circ \tilde{j}$$

$$= i^* \circ \bar{\varepsilon}_* \circ \tilde{j}_* \circ \pi_s^* \Omega^1_{\bar{\mathcal{B}}_s} \qquad \text{base change}$$

$$= \varepsilon_* \circ \bar{i}^* \circ \tilde{j}_* \circ \pi_s^* \Omega^1_{\bar{\mathcal{B}}_s} \qquad \bar{i} = \tilde{j} \circ i_s.$$

$$= \varepsilon_* i_s^* \tilde{j}^* \tilde{j}_* \pi_s^* \Omega^1_{\overline{\mathcal{B}}_s}$$

$0 \to T^*_{\overline{\mathcal{B}}_s}\mathcal{B} \xrightarrow{\tilde{j}} T^*\mathcal{B}|_{\overline{\mathcal{B}}_s} \to T^*\overline{\mathcal{B}}_s \to 0$

relative tangent bundle of $\tilde{j}$ is

$\pi_s^* T^* \overline{\mathcal{B}}_s$

$\tilde{j}^* \tilde{j}_*(-) = \lambda(\pi_s^*(T\overline{\mathcal{B}}_s)) \otimes -$

$$= \varepsilon_* i_s^* \pi_s^* (\lambda(T\overline{\mathcal{B}}_s) \otimes \Omega^1_{\overline{\mathcal{B}}_s})$$

$i_s^* \pi_s^* = \mathrm{Id}.$

$\lambda(T\overline{\mathcal{B}}_s) = \mathcal{O}_{\overline{\mathcal{B}}_s} - q^{-1} T\overline{\mathcal{B}}_s$

char. of $\mathbb{C}^*$ on $T\overline{\mathcal{B}}_s$

$$= \varepsilon_* (\mathcal{O}_{\overline{\mathcal{B}}_s} - q^{-1} T\overline{\mathcal{B}}_s) \otimes \Omega^1_{\overline{\mathcal{B}}_s})$$

$$= \varepsilon_* (\Omega^1_{\overline{\mathcal{B}}_s} - q^{-1} \mathcal{O}_{\overline{\mathcal{B}}_s}).$$

Thus,

$$\mathrm{res} \circ \bar{i}^* \circ \bar{j}_* (q\, Q_s)$$

$$= i^* j_* \circ \mathrm{res}(q\, Q_s)$$

$$= i^* j_* \pi_s^* (q\, \Omega^1_{\overline{\mathcal{B}}_s})$$

$$= \varepsilon_* (q\, \Omega^1_{\overline{\mathcal{B}}_s} - \mathcal{O}_{\overline{\mathcal{B}}_s}) \in K^{T \times \mathbb{C}^*}(\mathcal{B}) \overset{\mathrm{res}}{\simeq} K^{G \times \mathbb{C}^*}(\mathcal{B} \times \mathcal{B})$$

Now let's compute $\bar{i}^* \circ \bar{j}_* (q\, Q_s) = ?$

Recall $\mathrm{res}: K^{G \times \mathbb{C}^*}(\mathcal{B} \times \mathcal{B}) \simeq K^{G \times \mathbb{C}^*}(G \times_B \mathcal{B}) \simeq K^{T \times \mathbb{C}^*}(\mathcal{B})$

$\downarrow$ G/B.

restriction to the fiber over $\{b\}$

$$\begin{tikzcd} \{b\}\times\overline{B}_s \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{Y}_s \arrow[d, "pr_1"] \\ \{b\} \arrow[r, hook] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{aligned} \mathcal{O}_{\overline{B}_s} &= res(\mathcal{O}_{\overline{Y}_s}) \\ \Omega^1_{\overline{B}_s} &= res(\Omega^1_{\overline{Y}_s/B,\, pr_1}) \end{aligned}$$

$$\Longrightarrow \quad \bar{i}^* \circ \bar{j}_* (q Q_s) = q \cdot \Omega^1_{\overline{Y}_s/B,\, pr_1} - \mathcal{O}_{\overline{Y}_s} \in K^{G\times\mathbb{C}^*}(B\times B).$$

$\square$

Now we compare with Lusztig's action.

$\forall [F] \in K^G(G/B)$. $\quad \pi_s : G/B \to G/P_s$

$$T_s([F]) = \pi_s^* \pi_{s*}[F] - [F] - q\, \pi_s^* \pi_{s*} \big([F] \otimes [\overset{\Omega^1_{\pi_s}}{\overset{\|}{\Omega^1_s}}] \big)$$

$$\begin{tikzcd} \overline{Y}_s \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\pi_s"] \\ B \arrow[r, "\pi_s"'] & G/P_s \end{tikzcd}$$

$$\pi_s^* \pi_{s*}[F] = pr_{1*}(pr_2^* F \otimes \mathcal{O}_{\overline{Y}_s}) = \mathcal{O}_{\overline{Y}_s} * F$$

$$\begin{aligned} &\pi_s^* \pi_{s*}(F \otimes \Omega^1_s) \\ &= pr_{1*}\, pr_2^* (F \otimes \Omega^1_s) \\ &= pr_{1*}(pr_2^* F \otimes \Omega^1_{\overline{Y}_s/B,\, pr_1}) \\ &= \Omega^1_{\overline{Y}_s/B} * F \end{aligned}$$

Thus, Lusztig sends $-T_s-1$ to $q\,\Omega^1_{\overline{Y_s}/\mathcal{B}} - \mathcal{O}_{\overline{Y_s}} \in K^{G\times\mathbb{C}^*}(\mathcal{B}\times\mathcal{B})$.

$$\begin{gathered} q\,\Omega^1_{\overline{Y_s}/\mathcal{B}} - \mathcal{O}_{\overline{Y_s}} \\ \parallel \\ \bar{i}^*\bar{j}_*(q\,Q_s) \\ \parallel \\ \Theta(-T_s-1). \end{gathered}$$

Thus, we restricted to $\mathcal{B}\times\mathcal{B}$,

the map Θ = Lusztig's

This finishes the proof of step 2

Thus, $\psi_1=\psi_3=\psi_2$.

This concludes the proof of claim II.