§ Main results

Recall our goal is to classify all finite dim'l irreps of $\mathbb{H}$.

- Reduce to finite dim'l alg.

Affine Hecke alg. $\mathbb{H}$

$$\text{Center } Z(\mathbb{H}) = R(T)^W[q, q^{-1}].$$

$\mathbb{H}$ has basis $\{T_w e^\lambda \mid w \in W, \lambda \in P\}$, countable dimension.

Schur's lemma gives

<u>Lemma</u> For any simple $\mathbb{H}$-mod M, $Z(\mathbb{H})$ acts by scalar.

Thus, $\exists$ alg homomorphism $\chi : Z(\mathbb{H}) \to \mathbb{C}$, st.

$$Z(\mathbb{H}) \to \mathrm{End}(M) \quad \text{is} \quad z \mapsto \chi(z)\, \mathrm{Id}.$$

Since $Z(\mathbb{H}) \simeq R(T)^W[q, q^{-1}]$,

such χ corresponds to a semisimple element $a = (s, t) \in G \times \mathbb{C}^*$,

s.t. $\chi(z) = z(a) \quad \forall\, z \in Z(\mathbb{H})$.

Denote $\chi_a : Z(\mathbb{H}) \longrightarrow \mathbb{C}_a$

$z \longmapsto z(a)$

Define the specialized affine Hecke alg $\mathbb{H}_a := \mathbb{C}_a \otimes_{Z(\mathbb{H})} \mathbb{H}$.

Hence, we only need to classify irreps of $\mathbb{H}_a$.

By definition, $\mathbb{H}_a$ has dim $(\# W)^2$.

- Geometric interpretation of $\mathbb{H}_a$

$a = (s,t) \in G \times \mathbb{C}^*$ semisimple.

$\tilde{\mathcal{N}}^a$, $\mathcal{N}^a$, Z^a a-fixed points $\tilde{\mathcal{N}}^a$ is smooth

$Z^a = \tilde{\mathcal{N}}^a \times_{\mathcal{N}^a} \tilde{\mathcal{N}}^a$.

<u>Prop</u>: $\exists$ alg. isomorphism

$$\mathbb{H}_a \simeq H_*(Z^a, \mathbb{C}).$$ BM homology.

pf. $A = \langle a \rangle \subseteq G \times \mathbb{C}^*$ subgp generated by a.

Then we have

$$\mathbb{H}_a \simeq \mathbb{C}_a \otimes_{Z(\mathbb{H})} \mathbb{H}$$ definition

$$\simeq \mathbb{C}_a \otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(Z)$$ Kazhdan-Lusztig, Ginzburg

$$\simeq \mathbb{C}_a \otimes_{R(A)} K^A(Z)$$ $K^A(Z) \simeq R(A) \otimes_{R(G\times\mathbb{C}^*)} K^{G\times\mathbb{C}^*}(Z)$ 6.2 (6)

$$\overset{r_a}{\simeq} \mathbb{C}_a \otimes_{R(A)} K^A(Z^A)$$ localization, $r_a \simeq (\lambda_{\tilde{N}_a} \boxtimes 1) \cdot res_a$

Thm 5.11.10 invertible, isomorphism

$$\overset{ev}{\simeq} K_{\mathbb{C}}(Z^A)$$

$$\overset{RR}{\simeq} H_*(Z^A, \mathbb{C})$$ $RR = (1 \boxtimes Td_{\tilde{N}_a}) \cup ch_*$

Bivariant Riemann-Roch. Thm 5.11.11.

$$= H_*(Z^a, \mathbb{C})$$ $1 \boxtimes Td_{\tilde{N}_a}$ invertible, ch_* isomorphism

$\Rightarrow$ RR is an isomorphism.

r_a and RR preserve the convolution alg structures $\square$

- Standard modules

$a = (s, t) \in G \times \mathbb{C}^*$,

$\mathcal{N}^a = \{ x \in \mathcal{N} \mid s x s^{-1} = t \cdot x \}$.

$$\begin{array}{c} \tilde{\mathcal{N}}^a = \{ (x, b) \in \mathcal{N}^a \times \mathcal{B}^a \mid x \in b \} \\ \big\downarrow \mu \\ \mathcal{N}^a \end{array}$$

$x \in \mathcal{N}^a$, $\quad \mu^{-1}(x) = \mathcal{B}^s_x = \{ b \in \mathcal{B} \mid b \in \mathcal{B}^a,\ x \in b \} \subseteq \mathcal{B}$.

Since $s x s^{-1} = t x$, $\exp(z \cdot x)$ and s generate a solvable subgp of G

$\Rightarrow \mathcal{B}^s_x$ is non-empty.

By convolution $H_*(Z^a) \curvearrowright H_*(\mathcal{B}^s_x)$

$G(s, x) =$ simultaneous centralizer in G of s and x.

$C(s, x) = G(s, x) / G(s, x)^\circ$

- $C(s, x) \curvearrowright H_*(\mathcal{B}^s_x)$, commutes with the $H_*(Z^a)$-action.

<u>Def</u>. $C(s, x)^\wedge = \{ \text{simple } C(s, x)\text{-modules in } H_*(\mathcal{B}^s_x) \} / \simeq$

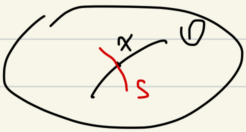
$\forall \psi \in C(s, x)^\wedge$, $\quad K_{a, x, \psi} = \mathrm{Hom}_{C(s, x)}(\psi, H_*(\mathcal{B}^s_x))$

It's called the standard $H_*(Z^a)$-module.

- costandard modules & Simple modules

$x \in \mathcal{N}^a$, $\mathbb{O} = G(s)\cdot x \subseteq \mathcal{N}^a$. $G(s)$-orbit of x.

S = local transverse slice to $\mathbb{O}$ at x. see Def 3.2.19 [CG]



$\mathcal{N}^a$

$\tilde{S} := \mu^{-1}(S) \subseteq \tilde{\mathcal{N}}^a$, $\mathcal{B}_x^s$ is a homotopy retract of $\tilde{S}$.

$\exists$ commuting actions of $H_*(Z^a)$ and $C(s,x)$ on $H_*(\tilde{S})$

<u>Def</u>: costandard $H_*(Z^a)$-module.

$$\mathrm{Hom}_{C(s,x)}(\psi, H_*(\tilde{S})),$$

$$\mathcal{B}_x^s \hookrightarrow \tilde{S} \rightsquigarrow H_*(\mathcal{B}_x^s) \xrightarrow{\varphi} H_*(\tilde{S})$$

Def. $L_{a,x} := \mathrm{Im}\,\varphi.$

$$= \bigoplus_{\psi \in C(s,x)^\wedge} L_{a,x,\psi} \otimes \psi.$$

Thus, $L_{a,x,\psi}$ = Image of Standard module in the co-standard mod

Prop. Assume t is not a root of unity, then $L_{a,x,\psi} \neq 0$

Let $M := \{a=(s,t) \in G \times \mathbb{C}^*,\ x \in N^a,\ \psi \in C(s,x)^\wedge \mid s \text{ is semisimple}\} / \mathrm{Ad}\,G$

Here, G acts on s by conjugation.

(x,ψ) and (x',ψ') are $G(s)$-conjugate, if $\exists\, g \in G(s)$, s.t.

$x' = gxg^{-1}$, and conjugation by g intertwines $C(s,x)$-module ψ

and $C(s,x')$-module ψ'.

Main theorem (Deligne-Langlands-Lusztig conjecture,

Kazhdan-Lusztig, Ginzburg theorem).

Assume t is not a root of unity, then $\{L_{a,x,\psi}\}_{(a,x,\psi) \in M}$ is a complete list of simple $\mathbb{H}$-modules, such that q acts by mult. by $t \in \mathbb{C}^*$.

Recall.

Deligne - Langlands - Lusztig.

$$\left\{\text{finite dim'l irrep of } \mathcal{H}_{aff}\right\}/_{\simeq} \xleftrightarrow{1:1} \left\{(s,x,\psi)\ \middle|\ \begin{array}{l} s\in \hat{G}(\mathbb{C}) \text{ s.s.},\ x\in\mathcal{N}, \\ sxs^{-1}=qx,\ \psi\in\hat{C}(s,x)\end{array}\right\}/_{\text{conj}}$$

$$sxs^{-1}=qx \iff x\in\mathcal{N}^{(s,t)}=\mathcal{N}^{a}.$$

Remarks: 1) $a=(s,t)\in G\times\mathbb{C}^{*}$ semisimple can be thought of "central characters" of the corresponding simple $\mathbb{H}$-modules.

2) Kazhdan and Lusztig proved that $\mathcal{N}^{a}$ is a finite union of $G(s)$-orbits. Thus, there are only finitely many simple $\mathbb{H}$-modules with a fixed central character.

3) the proof uses sheaf-theoretic methods