§. Sheaf-theoretic analysis of the simple modules

Notation: Given two graded vector space, write $V \doteq W$ for a linear isomorphism that does not preserve gradings.

Also use $\doteq$ to denote quasi-isomorphism that holds up to shifts.

$\mu: M \to N$ projective, M smooth. $m = \dim_{\mathbb{C}} M$.

decomposition thm, $\mu_* \mathbb{C}_M[m] = \bigoplus_{\substack{\phi = (N_\phi, \chi_\phi) \\ k}} L_\phi(k) \otimes IC_\phi[k]$,

Let $L_\phi = \bigoplus_k L_\phi(k)$

then $\mu_* \mathbb{C}_M[m] \doteq \bigoplus_\phi L_\phi \otimes IC_\phi$.

$x \in N$, $M_x = \mu^{-1}(x)$

By base change, $H_*(M_x) \cong H^{m-*}(i_x^! \mathbb{C}_M[m])$.

and $H^*(M_x) \cong H^{*-m}(i_x^* \mathbb{C}_M[m])$.

Let $U \subseteq N$ be a small enough open nbd. of x in N, s.t.

$\tilde{i}: M_x \hookrightarrow \tilde{U} := \mu^{-1}(U)$ is a homotopy equivalence.

Thus, $H^{m+*}(M_x) \overset{\tilde{i}^*}{\simeq} H^{m+*}(\tilde{u}) \underset{\text{duality}}{\overset{\text{Poincaré}}{\simeq}} H_{m-*}(\tilde{u})$

Also, $H^{m+*}(M_x) = H^*(i_x^* \mu_* \mathbb{C}_M[m])$

$$\doteq \bigoplus_{\varphi} L_\varphi \otimes H^*(i_x^* IC_\varphi).$$

Now assume $\{x\} = N_0$ is a one-point stratum in the stratification of N.

<u>Prop</u>: consider $\tilde{i}_* : H_*(M_x) \to H_*(\tilde{u})$, then

Image$(\tilde{i}_*) = L_x$, viewed as a subspace of $H_*(\tilde{u}) \doteq H^*(M_x)$

pf. First,

$$\begin{tikzcd}
H_{m-*}(M_x) \arrow[rrr, "\sim"] \arrow[d, "\tilde{i}_*"] & & & H^*(i_x^! \mu_* \mathbb{C}_M[m]) \arrow[d, "\varphi"] \\
H_{m-*}(\tilde{u}) \arrow[r, "\simeq", "\text{Poincaré}"'] & H^{m+*}(\tilde{u}) \arrow[r, "\tilde{i}^*", "\simeq"'] & H^{m+*}(M_x) \arrow[r, "\simeq"] & H^*(i_x^* \mu_* \mathbb{C}_M[m])
\end{tikzcd}$$

φ is induced by the natural homomorphism

$i_x^! \to i_x^*$ $\quad$ ($\Leftarrow$ Apply i^* to $i_! i^! \to \mathrm{id}$)

By the decomposition theorem,

$$\varphi : H^*(i_x^! \, \mu_* \mathbb{C}_M[m]) \longrightarrow H^*(i_x^* \, \mu_* \mathbb{C}_M[m])$$

$$= \bigoplus_{\phi} L_\phi \otimes \left(H^*(i_x^! IC_\phi) \to H^*(i_x^* IC_\phi) \right)$$

the prop follows from the following Lemma $\square$

<u>Lemma</u>: $X/\mathbb{C}$, $x \in X$, $Y \subseteq X$ locally closed. $i_x : \{x\} \hookrightarrow X$.

Then the canonical map $i_x^! IC(Y, \mathcal{L}) \to i_x^* IC(Y, \mathcal{L})$ vanishes

unless $Y = \{x\}$, in which case, it is a quasi-isomorphism.

pf: Assume $Y \neq \{x\}$, $x \in \bar{Y}$

Then $\dim Y > 0$. $\mathcal{H}^j(IC(Y, \mathcal{L})) = 0 \quad \forall j \geq 0$

$$\Rightarrow H^j(i_x^* IC(Y, \mathcal{L})) = 0 \quad \forall j \geq 0.$$

On the other hand,

$$H^j(i_x^! IC(Y, \mathcal{L})) = \left(H^{-j}(i_x^* IC(Y, \mathcal{L}^*)) \right)^* = 0 \quad \forall j \leq 0.$$

Hence, there is no nonzero map $H^*(i_x^! IC(Y, \mathcal{L})) \to H^*(i_x^* IC(Y, \mathcal{L}))$

If $Y=\{x\}$, $H^0(i_x^! IC(Y,\mathcal{L})) \simeq H^0(i_x^* IC(Y,\mathcal{L}))$ $\square$

<u>Rmk</u>. We also know the Kernel of $\tilde{i}_*: H_*(M_x) \to H_*(\tilde{u})$

$\langle -,- \rangle^{\tilde{u}}: H_{m+*}(M_x) \times H_{m-*}(M_x) \xrightarrow{\cap} \mathbb{C}$ intersecting pairing $M_x \subseteq \tilde{u}$.

$\ker \tilde{i}_* =$ radical of $\langle -,- \rangle^{\tilde{u}}$. (require M_x conn)

Equivariant case:

$\mu: M \to N$ G-equiv. proj, M smooth. $N = \coprod \mathbb{O}$ (G-orbits)

$x \in \mathbb{O}_x$,

Take a local transverse slice S to $\mathbb{O}_x$ at x.

$$\begin{tikzcd} \tilde{S} \arrow[r, hook] \arrow[d, "\mu"] \arrow[dr, phantom, "\square"] & M \arrow[d, "\mu"] \\ S \arrow[r, hook] & N \end{tikzcd}$$

shrink S if necessary, s.t.

$M_x \hookrightarrow \tilde{S}$ is a homotopy equivalence.

Then for the restriction $\mu: \tilde{S} \to S, = \coprod_{U} (S \cap U)$

$\{x\}$ is a one-point stratum for S.

Thus, we can apply the previous result. (take $U = S$).

Moreover, $H_*(M_x)$ and $H_*(\tilde{S})$ carry actions of $G(x)/G(x)^\circ$

<u>Prop</u>. ψ be an irrep of $G(x)/G(x)^\circ$. $\phi = (U_x, \psi)$.

Then $L_\phi \simeq \mathrm{Im}\left(H_*(M_x)_\psi \to H_*(\tilde{S})_\psi \right)$. as vector spaces.

Affine Hecke alg case

$a = (s,t) \in G \times \mathbb{C}^*$, $\quad M = \widetilde{\mathcal{N}}^a$, $N = \mathcal{N}^a$,

$$\begin{array}{ccc} M & = & \widetilde{\mathcal{N}}^a \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & = & \mathcal{N}^a = \bigsqcup_{G(s)\text{-orbits}} \mathbb{O} \end{array} \qquad G(s) \text{ acts.}$$

equiv. decomposition:

$$\mu_* \mathbb{C}_{\widetilde{\mathcal{N}}^a}[-] = \bigoplus_{\substack{i \in \mathbb{Z} \\ \phi = (\mathbb{O}, \psi)}} L_\phi(i) \otimes IC_\phi[i]$$

$x \in \mathbb{O} \subseteq \mathcal{N}^a$, $S \subseteq \mathcal{N}^a$ local transverse slice.

$\psi \in C(S,x)^\wedge$.

Recall the simple module

$$L_{a,x,\psi} := \mathrm{Im}\left(H_*(\mathcal{B}_x^s)_\psi \to H_*(\widetilde{S})_\psi\right).$$

$\mathcal{B}_x^s = \mu^{-1}(x)$ Thus

Prop. The simple module $L_{a,x,\psi} \simeq L_\phi := \bigoplus L_\phi(i)$

where $\phi = (\mathbb{O}, \psi)$.

Rmk: this gives the sheaf-theoretic interpretation of simple modules

§ Compatibility of actions

$\mu : M \to N$ M smooth, μ proper

$Z := M \times_N M$.

Recall we have proved:

1) $H_*(Z) \simeq \mathrm{Ext}^*_{D^b_c(N)}(\mu_* \mathbb{C}_M[m], \mu_* \mathbb{C}_M[m])$ as alg. isomorphism

2) $\mu_* \mathbb{C}_M[m] \doteq \bigoplus_\phi L_\phi \otimes IC_\phi$.

$$H_*(Z) \simeq \mathrm{Ext}^*(\mu_* \mathbb{C}_M[m], \mu_* \mathbb{C}_M[m])$$

$$= \bigoplus_\phi \mathrm{End}\, L_\phi \oplus \Big(\bigoplus_{\substack{\varphi, \psi \\ k > 0}} \mathrm{Hom}_{\mathbb{C}}(L_\varphi, L_\psi) \otimes \mathrm{Ext}^k(IC_\varphi, IC_\psi) \Big)$$

$$\twoheadrightarrow \bigoplus_\phi \mathrm{End}\, L_\phi.$$

$\{L_\phi\}_\phi$ is a complete list of isomorphism classes of simple $H_*(Z)$-modules.

Recall our goal is to classify all the simple $\mathbb{H}$-modules, which reduces to classify all simple $\mathbb{H}_a$-modules

Moreover, $\mathbb{H}_a \simeq H_*(Z^a)$ as algebras.

Thus, if we apply this to $\mu: \tilde{N}^a \to N^a$

$$\mu_* \mathbb{C}_{\tilde{N}^a}[-] = \bigoplus_{\phi} L_\phi \otimes IC_\phi.$$

We get $\{L_\phi\}_\phi$ is a complete list of isomorphism classes of $H_*(Z^a) \simeq \mathbb{H}_a$ modules.

We already proved the "simple" module

$$L_{a,x,\psi} \simeq L_\phi \quad \text{for } \phi = (\mathcal{O}_x, \psi) \text{ as vector spaces.}$$

Thus, we only need to prove

$$L_{a,x,\psi} \simeq L_\phi \quad \text{as } H_*(Z^a)\text{-modules}$$

We proved this in the general setting. $\mu: M \to N$.

- $L_\phi \simeq \mathrm{Im}(H_*(M_x)_\psi \to H_*(\tilde{S})_\psi)$ as $H_*(Z)$-modules.

$\mu: M \to N$, $Y \subseteq N$ locally closed

$A^\cdot \in D^b(N)$, $\forall u \in \mathrm{Ext}^k_D(A, A) = \mathrm{Hom}_D(A, A[k])$.

$u: A \to A[k]$

Apply $H^\cdot(Y, i^*(-))$ to this, we get

$H(Y, i^*(A)) \to H^{\cdot + k}(Y, i^*(A))$

$\rightsquigarrow \mathrm{Ext}^\cdot_D(A, A) \circlearrowright H^\cdot(Y, i^* A)$

Similarly, $\mathrm{Ext}^\cdot_D(A, A) \circlearrowright H^\cdot(Y, i^! A)$

Let $\mathbb{L} := \mu_* \mathbb{C}_M[m]$, $Y = \{x\}$,

$\rightsquigarrow \mathrm{Ext}^\cdot_{D^b(N)}(\mathbb{L}, \mathbb{L}) \circlearrowright H^*(i_x^* \mathbb{L}) \doteq H^*(M_x)$

$H^*(i_x^! \mathbb{L}) \doteq H_*(M_x)$.

Moreover, same argument as in the proof of

$\mathrm{Ext}^\cdot_D(\mathbb{L}, \mathbb{L}) \simeq H_*(Z)$ as algebras gives

Prop:

$$\begin{array}{ccc} H_*(Z)\times H_*(M_x) & \xrightarrow{\text{convolution}} & H_*(M_x) \\ \| & \circlearrowleft & \| \\ \mathrm{Ext}^{\cdot}_D(\mathbb{L},\mathbb{L})\times H^*(i_x^!\mathbb{L}) & \xrightarrow[\text{Yoneda}]{} & H^*(i_x^!\mathbb{L}) \end{array}$$

Similar result hold for cohomology.

Equivariant case:

$\mu: M\to N=\sqcup\mathbb{O}$, let $\mathbb{L}:=\mu_*\mathbb{C}_M[\dim]$

$x\in\mathbb{O}_x$, $\psi\in$ irrep of $G(x)/G(x)^{\circ}$. S local transverse slice to $\mathbb{O}$ at x.

$L_{x,\psi}:=\mathrm{Im}(H_*(M_x)_\psi\to H_*(\hat{S})_\psi)$

$\phi=(\mathbb{O},\psi)$

$L_\phi=$ multiplicity space of IC_ϕ in $\mathbb{L}$.

Recall $H_*(Z)\cong\mathrm{Ext}(\mathbb{L},\mathbb{L})$

$\twoheadrightarrow \mathrm{End}(\mathbb{L},\mathbb{L})=\bigoplus_\phi \mathrm{End}\,L_\phi\to\mathrm{End}\,L_\phi$

$\rightsquigarrow \mathrm{Ext}^{\cdot}(\mathbb{L},\mathbb{L})\curvearrowright L_\phi$

Recall we have proved $L_{\chi,\psi} \simeq L_\phi$ as v.s.

Lemma: $L_\phi \simeq L_{\chi,\psi}$ is an isomorphism of $\mathrm{Ext}^\cdot(\mathbb{L},\mathbb{L})$-modules
(8.6.17)

rough idea: use property of IC's to check

the $\mathrm{Ext}^\cdot(\mathbb{L},\mathbb{L})$ action on $L_{\chi,\psi}$ factor through

$$\mathrm{Ext}^\cdot(\mathbb{L},\mathbb{L}) \twoheadrightarrow \mathrm{End}(\mathbb{L}).$$

$\square$

Finally, we get.

Prop: The convolution action of $H_*(Z)$ on $L_{\chi,\psi}$

is the same as the $\mathrm{Ext}^\cdot(\mathbb{L},\mathbb{L})$-action on L_ϕ.

pf. the above Lemma shows

$\mathrm{Ext}^\cdot(\mathbb{L},\mathbb{L})$ actions on L_ϕ and $L_{\chi,\psi}$ are the same.

the previous Lemma shows

$H_*(Z)$ action on $L_{\chi,\psi}$ $=$ $\mathrm{Ext}^\cdot(\mathbb{L},\mathbb{L})$-action on $L_{\chi,\psi}$ $\square$.

Thm Every $H_*(2)$-module $L_{x,\psi}$ is simple, if non-zero.

Furthermore, any simple $H_*(2)$-module is isomorphic to $L_{x,\psi}$ for some pair (x, ψ)

This theorem, combined with a non-vanishing results, finish the proof of the Deligne-Langlands-Lusztig conjecture

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- A p-adic analogue of the Kazhdan-Lusztig formula.

Fix two parameters $\psi = (\mathcal{O}_\psi, \chi_\psi)$ and $\phi = (\mathcal{O}_\phi, \chi_\phi)$

choose $x \in \mathcal{O}_\psi$, $i_x : \{x\} \hookrightarrow N$.

Thm The multiplicity of the simple $H_*(2)$-module L_ϕ in the composition series of the $H_*(2)$-mod $H_*(M_x)_\psi$ is

$$[H_*(M_x)_\psi : L_\phi] = \sum_k \dim H^k(i_x^! IC_\phi)_\psi.$$

taking ψ-isotypical component

pf: $H_*(M_x) \doteq H^*(i_x^! \mathbb{L})$

$$\doteq \bigoplus_{\phi'} L_{\phi'} \otimes H^*(i_x^! IC_{\phi'})$$

By the Yoneda product,

$$\mathrm{Ext}^k(\mathbb{L}, \mathbb{L}) : \bigoplus_{\phi'} L_{\phi'} \otimes H^j(i_x^! IC_{\phi'}) \to \bigoplus_{\phi'} L_{\phi'} \otimes H^{j+k}(i_x^! IC_{\phi'})$$

$$\Rightarrow F^p H^*(i_x^! \mathbb{L}) := \bigoplus_{j \geq p} \left(\bigoplus_{\phi'} L_{\phi'} \otimes H^j(i_x^! IC_{\phi'}) \right)$$

is $\mathrm{Ext}^*(\mathbb{L}, \mathbb{L})$-stable.

$\rightsquigarrow$ a decreasing filtration for $H_*(M_x)$

$gr^F H^*(i_x^! \mathbb{L})$ = the ass. graded

Hence, $\mathrm{Ext}^*(\mathbb{L}, \mathbb{L})$ action on $gr^F H^*(i_x^! \mathbb{L})$ factors through

$$\mathrm{Ext}^*(\mathbb{L}, \mathbb{L}) \twoheadrightarrow \mathrm{End}\,\mathbb{L}.$$

As $\mathrm{End}\,\mathbb{L}$-modules,

$$gr^F H^*(i_x^! \mathbb{L}) \simeq \bigoplus_{\phi'} L_{\phi'} \otimes H^*(i_x^! IC_{\phi'})$$

Taking ψ-isotypical component,

$$\mathrm{gr}^F H^*(i_x^! \mathcal{L})_\psi \simeq \bigoplus_{\psi'} L_{\psi'} \otimes H^*(i_x^! IC_{\psi'})_\psi$$

Since $\mathrm{End}(\mathcal{L})$ is semisimple,

$$[\mathrm{gr}^F H^*(i_x^! \mathcal{L})_\psi : L_\psi] = \sum \dim H^k(i_x^! IC_\psi)_\psi$$

Replacing a module by its ass graded doesn't affect the multiplicities, we get the result. $\square$

A non-vanishing result.

We assume t is not a root of unity

Thm: the simple module $L_{a,x,\psi} \neq 0$ for any $x \in \mathcal{N}^a$,

and any irr. rep $\psi \in C(s,x)^\wedge$.

$a = (s,t) \in G \times \mathbb{C}^*$, $\mu : \tilde{\mathcal{N}}^a \to \mathcal{N}^a$, $\quad \mathbb{O} = G(s)\cdot x \in \mathcal{N}^a$

We need two lemmas.

Lemma 1 (8.8.1, Reeder).

$\exists$ $G(s)$-stable subset $\hat{\mathbb{O}}$ of $\tilde{\mathcal{N}}^a$, which is both open and closed in $\tilde{\mathcal{N}}^a$, s.t. $\mu(\hat{\mathbb{O}}) = \overline{\mathbb{O}}$.

Let $\hat{\mathcal{B}}_x^s = \hat{\mathbb{O}} \cap \mathcal{B}_x^s$, an open and closed $G(s,x)$-stable subset in $\mathcal{B}_x^s$.

Lemma 2 (8.8.2, Grojnowski)

Any simple $C(s,x)$-mod occuring in $H^*(\mathcal{B}_x^s)$ with non-zero multiplicity occurs in $H^*(\hat{\mathcal{B}}_x^s)$ with non-zero multiplicity.

Pf of the theorem:

$$\mu_* \mathbb{C}_{\tilde{\mathcal{N}}^a}[\dim] = \bigoplus_{i, \phi=(\mathcal{O}, \psi)} L_\phi(i) \otimes IC_\phi[-i]$$

Recall we already proved $L_{a,x,\psi} \simeq \bigoplus_i L_\phi(i) = L_\phi$

Thus, we only need to show that for any $\psi \in C(s,x)^\wedge$,

IC_ϕ occurs in the decomposition.

By lemma 1. $\hat{\mathcal{O}}$ is the union of several conn. components of $\tilde{\mathcal{N}}^a$

Thus, $\mu_* \mathbb{C}_{\tilde{\mathcal{N}}^a}[-] = \mu_* \mathbb{C}_{\hat{\mathcal{O}}}[-] \oplus A$

Moreover, $\mu_* \mathbb{C}_{\hat{\mathcal{O}}}[-] = \left(\bigoplus_{i,\psi} \hat{L}_\psi(i) \otimes IC(\mathcal{O}, \psi)[i] \right) \oplus B.$

where ψ runs over the set of irreps of $C(s,x)$, that occurs in

$H^*(\hat{\mathcal{B}}_x^s)$, B is supp on $\bar{\mathcal{O}} \setminus \mathcal{O}$.

Taking the stalk at $x \in \mathcal{O}$

$$H^*(\hat{\mathcal{B}}_x^s) \doteq \bigoplus_\psi \hat{L}_\psi \otimes \psi. \qquad \hat{L}_\psi = \bigoplus_i \hat{L}_\psi(i).$$

Thus, $\psi \in C(s,x)^\wedge \overset{\text{Lemma 2}}{\Longrightarrow} \psi$ occurs in $H^*(\hat{\mathcal{B}}_x^s)$

$\Rightarrow IC(\mathcal{O},\psi)$ occurs in $\mu_*\mathbb{C}_{\tilde{\mathfrak{g}}}[-]$

$\Rightarrow IC(\mathcal{O},\psi)$ occurs in $\mu_*\mathbb{C}_{\tilde{\mathcal{N}}^a}[-]$

$\Rightarrow L_{a,x,\psi} = L_\phi \neq 0$

$\square$

A natural continuation of the study is the categorification of the Kazhdan-Lusztig / Ginzburg isomorphism

$$K^{G\times\mathbb{C}^*}(Z) \simeq \mathbb{H} = \mathbb{C}_c[I\backslash \check{G}(F)/I],$$

see the discussion in the introduction of [CG].

This is done by Bezrukavnikov.

A good reference is Williamson's course notes, available on arxiv, Langlands correspondence and Bezrukavnikov's equivalence