1) the Steinberg variety.

<u>Def</u>. $Z := \tilde{N} \times_N \tilde{N}$

$$= \{ (x, \mathfrak{b}, \mathfrak{b}') \mid x \in N,\ \mathfrak{b}, \mathfrak{b}' \in \mathcal{B},\ x \in \mathfrak{b} \cap \mathfrak{b}' \}$$

$$\begin{tikzcd} & Z \arrow[dl, "\mu_Z"'] \arrow[dr, "\pi \times \pi"] & \\ N & & \mathcal{B} \times \mathcal{B} \end{tikzcd}$$

Sign convention: $T^*(X_1 \times X_2) \simeq T^*X_1 \times T^*X_2$

change sign in the T^*X_2 factor.

Hence $T^*_{\Delta X}(X \times X) = \Delta(T^*X)$

$\Delta X \subseteq X \times X$ diagonal

$$\tilde{N} \times \tilde{N} \simeq T^*\mathcal{B} \times T^*\mathcal{B} \overset{\text{sign}}{\simeq} T^*(\mathcal{B} \times \mathcal{B}).$$

$$(x_1, \mathfrak{b}_1, x_2, \mathfrak{b}_2) \longmapsto (x_1, \mathfrak{b}_1, -x_2, \mathfrak{b}_2)$$

<u>Prop</u>: 1) $Z = \bigsqcup_{w \in W} T^*_{Y_w}(\mathcal{B}\times\mathcal{B})$, where

$Y_w = G.(\mathfrak{b}, w.\mathfrak{b}) \subseteq \mathcal{B}\times\mathcal{B}$.

2) Irreducible components of Z are $\overline{T^*_{Y_w}(\mathcal{B}\times\mathcal{B})}$.

$\dim Z = 2\dim\mathcal{B} = \dim\widetilde{\mathcal{N}}$.

pf: 1) $(\mathfrak{b}_1, \mathfrak{b}_2) \in Y(w) \subseteq \mathcal{B}\times\mathcal{B}$.

$T_{(\mathfrak{b}_1,\mathfrak{b}_2)} Y(w) = \{(x \bmod \mathfrak{b}_1, x \bmod \mathfrak{b}_2) \mid x \in \mathfrak{g}\} \subseteq T_{\mathfrak{b}_1}\mathcal{B} \times T_{\mathfrak{b}_2}\mathcal{B}$

Hence, if $\alpha = (x_1, \mathfrak{b}_1, x_2, \mathfrak{b}_2) \in T^*\mathcal{B}\times T^*\mathcal{B} \subseteq \mathfrak{g}^*\times\mathcal{B}\times\mathfrak{g}^*\times\mathcal{B}$.

is annihilated by $T_{(\mathfrak{b}_1,\mathfrak{b}_2)} Y(w)$, then

$\langle x_1, x\rangle + \langle x_2, x\rangle = 0 \quad \forall\, x \in \mathfrak{g}$.

$\Rightarrow x_1 = -x_2$

$\Rightarrow \alpha = (x_1, \mathfrak{b}_1, -x_1, \mathfrak{b}_2) \in Z$. $\square$

2) follows from 1)

Recall $\mathfrak{g}$ is semisimple, $\mathfrak{g} \simeq \mathfrak{g}^*$

coadjoint orbits in $\mathfrak{g}^*$ $\rightsquigarrow$ adjoint orbit in $\mathfrak{g}$.

<u>Thm</u> (3.3.7). for any G-orbit $\mathcal{O} \subseteq \mathfrak{g}$, and any $x \in \mathcal{O} \cap \mathfrak{b}$, the set $\mathcal{O} \cap (x + \mathfrak{n})$ is a Lagrangian subvariety in $\mathcal{O}$.

$\mu_Z : Z \to \mathcal{N}$ $\qquad Z_{\mathcal{O}} := \mu_Z^{-1}(\mathcal{O})$

<u>Cor</u> each irr. comp. of $Z_{\mathcal{O}}$ has $\dim = \dim Z$.

<u>Pf</u>: $\tilde{\mathcal{O}} = \mu^{-1}(\mathcal{O}) \subseteq T^*\mathcal{B}$.

$Z_{\mathcal{O}} = \tilde{\mathcal{O}} \times_{\mathcal{O}} \tilde{\mathcal{O}}$

$\tilde{\mathcal{O}} \simeq G \times_B (\mathcal{O} \cap \mathfrak{n})$. $\Rightarrow$ irr. comp. of $\tilde{\mathcal{O}}$ has dim

$= \dim \mathcal{B} + \dim \mathcal{O} \cap \mathfrak{n} = \dim \mathcal{B} + \frac{1}{2} \dim \mathcal{O}$

(use the above theorem when $x \in \mathfrak{n} \cap \mathcal{O}$).

$\Rightarrow$ irr. comp. of $Z_{\mathcal{O}} = \tilde{\mathcal{O}} \times_{\mathcal{O}} \tilde{\mathcal{O}}$ has dim

$$= 2\dim \tilde{\mathcal{O}} - \dim \mathcal{O} = 2\dim \mathcal{B} = \dim Z \qquad \square$$

$x \in \mathcal{O}$, G_x = stabilizer of x, $\mathcal{O} \simeq G/G_x$

$\mathcal{B}_x = \mu^{-1}(x) \subseteq \mathcal{B}$,

then

$$\begin{tikzcd} \tilde{\mathcal{O}} \arrow[r, phantom, "\simeq"] \arrow[dr] & G \times_{G_x} \mathcal{B}_x \arrow[d] \\ & \mathcal{O} \simeq G/G_x \end{tikzcd}$$

$$Z_{\mathcal{O}} = \tilde{\mathcal{O}} \times_{\mathcal{O}} \tilde{\mathcal{O}} \simeq G \times_{G_x} (\mathcal{B}_x \times \mathcal{B}_x).$$

therefore, each irr. comp. of $Z_{\mathcal{O}}$ is of the form

$$G \times_{G_x} (\mathcal{B}_1 \times \mathcal{B}_2),$$

$\mathcal{B}_1, \mathcal{B}_2$ irr. comp. of $\mathcal{B}_x$.

$\Rightarrow \dim \mathcal{O} + \dim \mathcal{B}_1 + \dim \mathcal{B}_2 = \dim Z_{\mathcal{O}} = 2\dim \mathcal{B}$.

Cor (Spaltenstein).

1) All irr. comps of $\mathcal{B}_x$ have the same dim,

and $\frac{1}{2}\dim \mathcal{O} + \dim \mathcal{B}_x = \dim \mathcal{B}$.

2) $\mathcal{B}_x$ is connected.

pf: 2) follows from Zariski main theorem + $\mathcal{N}$ is normal (Kostant).

$\left(X' \xrightarrow{f} X \text{ proper, birational, } X \text{ normal} \right.$

$\left. \Rightarrow f^{-1}(x) \text{ is connected } \forall x \in X. \right)$. □

Let $C(x) = G_x / G_x^{\circ}$ be the group of connected components.

$G(x) \curvearrowright \mathcal{B}_x \Rightarrow C(x) \curvearrowright \{\mathcal{B}_x^{\alpha}\} =$ irr. comps of $\mathcal{B}_x^{\alpha}$.

Cor: irr. comps of $Z_{\mathcal{O}}$ is in bijection with the $C(x)$-orbits on pairs of comps of $\mathcal{B}_x$.

Cor: # G-orbits on $\mathcal{N}$ is finite.

pf: $Z = \bigsqcup_{\mathcal{O}} Z_{\mathcal{O}}$,

$Z_{\mathcal{O}}$ have the same dimension $\Rightarrow$ closure of an irr. comp of $Z_{\mathcal{O}}$ is an irr. comp of Z.

irr. comps of Z = $\# W$.

$\Rightarrow \#\{\mathcal{O}\}$ is finite. $\square$

2) Borel-Moore homology

X complex or real alg variety.

the Borel-Moore homology can be defined in the following equivalent ways:

ⓐ $\hat{X} = X \cup \{\infty\}$ one-point compactification of X

$$H_*^{BM}(X) := H_*(\hat{X}, \infty) \quad \text{relative homology.}$$

ⓑ $\bar{X}$ an arbitrary compactification of X, such that $(\bar{X}, \bar{X}\setminus X)$ is a CW-pair.

$$H_*^{BM}(X) = H_*(\bar{X}, \bar{X}\setminus X)$$

ⓒ Let $C_*^{BM}(X)$ = chain complex of infinite singular chains $\sum a_i \sigma_i$, σ_i a singular simplex, $a_i \in \mathbb{C}$, the sum is locally finite: for any compact set $D \subseteq X$, there are only finitely many non-zero coefficients a_i, such that $D \cap \operatorname{supp} \sigma_i \neq \emptyset$.

$$H_*^{BM}(X) \simeq H_*(C_*^{BM}(X), \partial)$$ ← usual boundary map

(c) Poincare duality.

M smooth, oriented manifold, $m = \dim_{\mathbb{R}} M$.

$X \subseteq M$ closed, has a closed neighborhood $U \subseteq M$ such that X is a proper deformation retract of U.

$$H_i^{BM}(X) = H^{m-i}(M, M \setminus X)$$

in particular, $H_i^{BM}(M) \simeq H^{m-i}(M)$

Rmk: $\exists$ sheaf-theoretic definition.

Notation: $H_i := H_i^{BM}$ $\qquad H_i^{ord}$ = ordinary homology

$$H_i^{ord}(M) \simeq H_c^{m-i}(M)$$

Proper pushforward:

$f: X \to Y$ proper (inverse image of compact is compact)

$\rightsquigarrow f_*: H_*(X) \to H_*(Y)$

by extending f to a $\bar{f}: \bar{X} = X \cup \{\infty\} \to \bar{Y} = Y \cup \{\infty\}$,

$\bar{f}(\infty) = \infty$, which is a continuous map.

Long exact sequence.

$$\underset{\text{closed}}{F} \hookrightarrow X \hookleftarrow U := X \setminus F$$

$$\rightsquigarrow \quad \cdots \to H_p(F) \to H_p(X) \to H_p(U) \to H_{p-1}(F) \to \cdots$$

Fundamental class.

if X is smooth, oriented manifold,

$\exists$ fundamental class $[X] \in H_m(X)$, $m = \dim_{\mathbb{R}} X$.

For an arbitrary (not necessarily smooth or compact) complex alg. variety X, $\exists$ fundamental class. It's construction is as follows:

(a) if X is irr. of real dim m, X^{reg} = Zariski open dense subset consisting of non-singular points of X.

$\Rightarrow \exists [X^{reg}] \in H_m(X^{reg})$.

Since $\dim_{\mathbb{R}}(X \setminus X^{reg}) \leq m-2$

$H_k(X \setminus X^{reg}) = 0$ for any $k > m-2$.

The long exact sequence for $X \setminus X^{reg} \hookrightarrow X \hookleftarrow X^{reg}$

shows $H_m(X) \xrightarrow{\sim} H_m(X^{reg})$

define $[X] :=$ preimage of $[X^{reg}] \in H_m(X^{reg})$

(b) If X has irr. comps. $X_1, X_2, \cdots, X_n$,

define $[X] := \sum [X_i]$.

Prop: Let X be a complex variety of $\dim_{\mathbb{R}} X = m$.
Let $X_1, \dots, X_n$ be the n-dim'l irr. comps of X. then
$[X_1], [X_2], \dots, [X_n]$ is a basis for $H_{top}(X) = H_m(X)$.

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intersection pairing.

M smooth oriented manifold, $Z_1, Z_2 \overset{\text{closed}}{\subseteq} M$

$$\cap : H_i(Z_1) \times H_j(Z_2) \to H_{i+j-m}(Z_1 \cap Z_2), \quad m = \dim_{\mathbb{R}} M$$

$$\Updownarrow$$

$$\cup : H^{m-i}(M, M \setminus Z_1) \times H^{m-j}(M, M \setminus Z_2) \to H^{2m-i-j}(M, (M \setminus Z_1) \cup (M \setminus Z_2))$$

Künneth formula

$$\boxtimes : H_*(M_1) \otimes H_*(M_2) \xrightarrow{\sim} H_*(M_1 \times M_2)$$

Smooth pullback.

For a trivial fibration $p: X \times F \to X$,

where F is smooth and oriented, $\dim_{\mathbb{R}} F = d$.

$\exists\ p^*: H_i(X) \to H_{i+d}(X \times F)$,

$$c \mapsto c \boxtimes [F].$$

In general, $p: \tilde{X} \to X$ locally trivial fibration with

fiber F (smooth and oriented).

$\exists\ p^*: H_i(X) \to H_{i+d}(\tilde{X})$,

and it has the above form when we restrict to

any open $U \subseteq X$, s.t. p is a trivial fibration.

$i: X \hookrightarrow \tilde{X}$ a continuous section of p.

can define Gysin pullback $i^*: H_i(\hat{X}) \to H_{i-d}(X)$.

such that $i^* \circ p^* = \mathrm{Id}$.

In the trivial fibration case $p: X \times F \to X$

$$H_*(\tilde{X}) \simeq H_*(X) \boxtimes H_*(F),$$

$i^*(c \boxtimes [F]) = c$,

$i^*(c \boxtimes \gamma) = 0$ if $\gamma \in H_{<d}(F)$.

Specialization map in Borel-Moore homology

(S, o) a smooth manifold with base point $o \in S$.

$S^* = S \setminus \{o\}$.

$\pi: Z \to S$, $Z_o = \pi^{-1}(o)$, $\forall\, s' \subseteq S$ $Z(s') := \pi^{-1}(s')$

Assume $\pi: Z(S^*) \to S^*$ is a locally trivial fibration

with possibly singular fiber. (Note π is not assumed to

be locally trivial near o).

we want to define a specialization map

$$\lim_{s\to 0} : H_*(Z(S^*)) \to H_{*-d}(S_0), \quad d = \dim_{\mathbb{R}} S.$$

Construction: choose an open nbd $(B,0)$ of 0 in S, diffeomorphic to $(\mathbb{R}^d, 0)$.

$\mathbb{R}^d_{\geq 0} := \mathbb{R}_{\geq 0} \times \mathbb{R}^{d-1}$, $\quad B_{\geq 0} \subseteq B$ the corresponding space.

$B_{\geq 0}$ is contractible,

Shrink B if necessarily, such that $\pi : Z(B_{>0}) \to B_{>0}$ is a trivial fibration with fiber F,

$I_{>0}$ (resp. I) $\subseteq B$ the corresponding space of $\mathbb{R}_{>0}$ (resp $\mathbb{R}_{\geq 0}$) in $\mathbb{R} \subseteq \mathbb{R} \times \mathbb{R}^{d-1} \simeq \mathbb{R}^d$.

Then

$$H_*(Z(S^*)) \xrightarrow[\text{to open}]{\text{restriction}} H_*(Z(B_{>0})) \xrightarrow[\sim]{\text{Künneth}} H_{*-d}(F) \otimes H_d(B_{>0})$$

$$\xrightarrow[\text{fundamental classes}]{\sim} H_{*-d}(F) \otimes H_1(I_{>0}) \xrightarrow[\text{Künneth}]{\sim} H_{*-d+1}(Z(I_{>0}))$$

$\xrightarrow{\partial} H_{*-d}(Z_0)$

$\uparrow$ long exact sequence from

$Z_0 \hookrightarrow Z(I_{\geq 0}) \hookleftarrow Z(I_{>0})$.