1) Convolution in Borel-Moore homology

Toy example of convolution product.

for a finite set M, let $\mathbb{C}(M)$ be the $\mathbb{C}$-valued functions on M.

Given finite sets M_1, M_2, M_3, define a convolution product

$$\mathbb{C}(M_1 \times M_2) \times \mathbb{C}(M_2 \times M_3) \longrightarrow \mathbb{C}(M_1 \times M_3)$$

by

$$f_{12} * f_{23}(m_1, m_3)$$

$$:= \sum_{m_2 \in M_2} f_{12}(m_1, m_2) \cdot f_{23}(m_2, m_3).$$

$\mathbb{C}(M_i \times M_j) \simeq \# M_i \times \# M_j$ matrices

$*$ = matrix product.

General Case.

M_1, M_2, M_3 connected oriented manifolds

$$Z_{12} \subseteq M_1 \times M_2, \qquad Z_{23} \subseteq M_2 \times M_3 \qquad \text{closed.}$$

$$Z_{12} \circ Z_{23} := \left\{ (m_1, m_3) \in M_1 \times M_3 \;\middle|\; \begin{array}{l} \exists\, m_2 \in M_2, \text{ s.t.} \\ (m_1, m_2) \in Z_{12},\ (m_2, m_3) \in Z_{23} \end{array} \right\}$$

Ex $f: M_1 \to M_2$, $g: M_2 \to M_3$ smooth maps,

$$\text{Graph}(f) \circ \text{Graph}(g) = \text{Graph}(g \circ f)$$

Let $p_{ij}: M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the projections

assume $p_{13}: p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3$ is proper.

By definition, $Z_{12} \circ Z_{23}$ = Image of the above map

$d = \dim_{\mathbb{R}} M_2$, we define a convolution in Borel-Moore homology as follows:

$$H_i(Z_{12}) \times H_j(Z_{23}) \longrightarrow H_{i+j-d}(Z_{12}\circ Z_{23})$$

$$(C_{12}, C_{23}) \longmapsto C_{12} * C_{23}$$

$$C_{12} * C_{23} := p_{13*}\left(p_{12}^* C_{12} \cap p_{23}^* C_{23}\right)$$

$$= p_{13*}\Big(\underset{\underset{H_{i+\dim_{\mathbb{R}} M_3}}{\rotatebox{90}{$\in$}}}{(C_{12}\boxtimes[M_3])} \cap \underset{\underset{H_{j+\dim_{\mathbb{R}} M_1}}{\rotatebox{90}{$\in$}}}{([M_1]\boxtimes C_{23})}\Big)$$

Ex 1) $M_1 = M_2 = M_3 = M$, $\quad Z_{12}, Z_{23} \subseteq \overset{\text{diagonal}}{M_\Delta} \subseteq M \times M$

$$Z_{12}\circ Z_{23} = Z_{12}\cap Z_{23}$$

$* =$ intersection product.

2) $M_1 = pt$, $f: M_2 \to M_3$ proper.

$Z_{12} = pt \times M_2$, $\quad Z_{23} = \mathrm{Graph}(f)$, $\quad Z_{12}\circ Z_{23} = \mathrm{Im} f \subseteq pt \times M_3$

Let $c \in H_*(M_2) = H_*(Z_{12})$.

$\Rightarrow \quad c * [\mathrm{Graph}(f)] = f_*(c)$.

3). $M_3 = pt$, $f: M_1 \to M_2$ smooth.

$Z_{12} = \mathrm{Graph}(f)$, $Z_{23} = M_2 \times pt$.

$Z_{12} \circ Z_{23} = M_1 \times pt = M_1$

$\Rightarrow \quad [\mathrm{Graph}(f)] * c = f^*(c)$.

Associativity of convolution.

M_1, M_2, M_3, M_4 as before. $Z_{34} \subseteq M_3 \times M_4$

$c_{ij} \in H_*(Z_{ij})$

Lemma: $(c_{12} * c_{23}) * c_{34} = c_{12} * (c_{23} * c_{34})$.

Pf: $p^{1234}_{123} : M_1 \times M_2 \times M_3 \times M_4 \to M_1 \times M_2 \times M_3$,

Similarly for p^{123}_{12}, ...

$$(C_{12} * C_{23}) * C_{34}$$

$$= p^{134}_{14*}\left(\left(p^{123}_{13*}\left(\left(C_{12} \boxtimes [M_3]\right) \cap \left([M_1] \boxtimes C_{23}\right)\right) \boxtimes [M_4]\right) \cap \left([M_1] \boxtimes C_{34}\right)\right)$$

$$= p^{134}_{14*}\left(p^{1234}_{134*}\left(\left(\left(C_{12} \boxtimes [M_3] \boxtimes [M_4]\right) \cap \left([M_1] \boxtimes C_{23} \boxtimes [M_4]\right)\right)\right) \cap \left([M_1] \boxtimes C_{34}\right)\right)$$

projection formula

$$= p^{134}_{14*}\, p^{1234}_{134*}\left(\left(\left(C_{12} \boxtimes [M_3] \boxtimes [M_4]\right) \cap \left([M_1] \boxtimes C_{23} \boxtimes [M_4]\right)\right) \cap \left([M_1] \boxtimes [M_2] \boxtimes C_{34}\right)\right)$$

$$[M_1] \boxtimes [M_2] \boxtimes C_{34} = p^{134*}_{134}\left([M_1] \boxtimes C_{34}\right)$$

$$= p^{1234}_{14*}\left(\_\_\_\_ \right)$$

$$= C_{12} * (C_{23} * C_{34}).$$

□

Specialization commutes with convolution

S base manifold, $f_i : M_i \to S$, $0 \in S$. $S^* = S \setminus \{0\}$

$f : Z \to S$, $Z^0 := f^{-1}(0)$, $Z^* := f^{-1}(S^*)$.

Assume: (1) $Z_{ij}^* \to S^*$ is a locally trivial fibration.

(2)
$$\begin{tikzcd} p_{12}^{-1}(Z_{12}^*) \cap p_{23}^{-1}(Z_{23}^*) \arrow[r, "p_{13}"] \arrow[dr] & Z_{13}^* \arrow[d] \\ & S^* \end{tikzcd}$$

the horizontal map is local trivial fibrations

<u>Prop</u>:
$$\begin{tikzcd} H_*(Z_{12}^*) \times H_*(Z_{23}^*) \arrow[r, "\lim_{t\to 0}"] \arrow[d, "\text{convolution}"] \arrow[dr, phantom, "\circlearrowright"] & H_*(Z_{12}^0) \times H_*(Z_{23}^0) \arrow[d] \\ H_*(Z_{13}^*) \arrow[r, "\lim_{t\to 0}"] & H_*(Z_{13}^0) \end{tikzcd}$$

Idea of pf: convolution = proper pushforwards + intersection pairing.

① Specialization commutes with intersection.

intersection pairing $\overset{\text{diagonal}}{\underset{\text{reduction.}}{=\!=\!=}}$ restriction with supports restriction with supports: $N \overset{i}{\hookrightarrow} M$ closed, $d = \dim_{\mathbb{R}} M - \dim_{\mathbb{R}} N$

$Z \subseteq M$ $\quad i^* : H_*(Z) \to H_{*-d}(Z \cap N)$

$$\Updownarrow$$

$$i^* : H^*(M, M \setminus Z) \to H^*(N, N \setminus (N \cap Z))$$

diagonal reduction

$Z, Z' \subseteq M,$ $\quad i_\Delta : M_\Delta \hookrightarrow M \times M$

$c \in H_*(Z),$ $\quad c' \in H_*(Z')$

$$c \cap c' = i_\Delta^* (c \boxtimes c')$$

---

relative version of restriction with supports:

$$\begin{tikzcd} N \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & M \arrow[d, "\pi"] & Z \arrow[l, hook'] \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

$i : (Z \cap N, Z_0 \cap N) \hookrightarrow (Z, Z_0)$

induces a morphism of the long-exact sequences

$\Rightarrow$ j^* commutes with the connecting morphism in the specialization map.

② Proper pushforward commutes with specialization by the same reason as above. $\square$

The convolution algebra.

M smooth manifold / $\mathbb{C}$. $\pi: M \to N$ proper.

$M_1 = M_2 = M_3 = M$, $Z = Z_{12} = Z_{23} = M \times_N M$.

$\Rightarrow Z \circ Z = Z$.

$\Rightarrow H_*(Z) \times H_*(Z) \xrightarrow{*} H(Z)$

<u>Cor</u>: $H_*(Z)$ has a natural structure of an associative alg with unit. the unit $= \Delta(M) \subseteq Z$.

---

choose $x \in N$. $M_x := \pi^{-1}(x)$

$M_1 = M_2 = M$, $M_3 = pt$, $Z = Z_{12} = M \times_N M$.

$Z_{23} = M_x \subset M \times \{pt\}$

$\Rightarrow Z_{12} \circ Z_{23} = M_x$

<u>Cor</u> $H_*(M_x)$ has a natural structure of a left

$H_*(Z)$-module under convolution.

The dimension property.

$\dim_{\mathbb{R}} M_i = d_i$. $\quad p = \frac{m_1+m_2}{2}$, $q = \frac{m_2+m_3}{2}$, $r = \frac{m_1+m_3}{2}$

$$H_p(Z_{12}) \times H_q(Z_{23}) \to H_r(Z_{12} \circ Z_{23})$$

∴ the middle dimension part is always preserved".

$M_1 = M_2 = M_3 = M$, $\dim_{\mathbb{R}} M = m$. $\quad Z = M \times_N M$.

$H(Z) := H_m(Z)$.

Cor: $H(Z)$ is a subalg of $H_*(Z)$.

Lem: If Irr. comps of Z are $\{Z_i \mid i \in I\}$, I finite.

and $\dim Z_i = \dim Z$.

then $\{[Z_i]\}$ is a basis for the convolution alg $H(Z)$.

<u>Cor</u>: The convolution action of $H(2) \subseteq H_*(2)$ on $H_*(M_\pi)$ is degree preserving, i.e. for any $j \geq 0$,

$$H(2) * H_j(M_\pi) \subseteq H_j(M_\pi).$$

2) Lagrangian Construction of the Weyl group.

Recall

$$
\begin{tikzcd}
& \tilde{\mathcal{N}} \arrow[r, hook] \arrow[dl, "\mu"'] & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
\mathcal{N} \arrow[r, hook] \arrow[dr, two heads] & \mathfrak{g} \arrow[dr, two heads] & & \mathfrak{h} \arrow[dl] \\
& 0 \arrow[r, hook] & \mathfrak{h}/\!/W &
\end{tikzcd}
$$

$$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$$

<u>Thm</u>: $H_{top}(Z) \simeq \mathbb{Q}[W]$ as ass. algs.

Example: $G = SL(2,\mathbb{C})$, $Z = T^*\mathbb{P}^1 \times_{\mathcal{N}} T^*\mathbb{P}^1$

irr. comps of Z are $T^*_{Y(id)}(\mathbb{P}^1\times\mathbb{P}^1) = \Delta(T^*\mathbb{P}^1) =: T^*_{id}$,

$\overline{T^*_{Y(s_\alpha)}(\mathbb{P}^1\times\mathbb{P}^1)} = \mathbb{P}^1\times\mathbb{P}^1 =: T^*_{s_\alpha}$

$$T^*_{s_\alpha} * T^*_{s_\alpha} = [\mathbb{P}^1\times\mathbb{P}^1] \cdot \int_{T^*\mathbb{P}^1} [\mathbb{P}^1]\cdot[\mathbb{P}^1] .$$

$$= -2\, T^*_{s_\alpha}$$

$T^*_{id} * T^*_{id} = T^*_{id}$, $\quad T^*_{id} * T^*_{s_\alpha} = T^*_{s_\alpha}$

Thus $\quad H(Z) \simeq \mathbb{Q}[W]$

$$T^*_{id} \longleftrightarrow id$$

$$T^*_{s_\alpha} + T^*_{id} \longleftrightarrow s_\alpha$$

representations ?

a) $x=0$, $\mu^{-1}(0) = \mathbb{P}^1 \subseteq T^*\mathbb{P}^1$, $H_{top}(\mathbb{P}^1) = \text{Span of } [\mathbb{P}^1]$

$$T^*\mathbb{P}^1 \times T^*\mathbb{P}^1 \times pt$$

$$\mathbb{P}^1 \times \mathbb{P}^1 \qquad \mathbb{P}^1$$

$\Rightarrow$ $[\mathbb{P}^1 \times \mathbb{P}^1] * [\mathbb{P}^1] = -2[\mathbb{P}^1]$

$\Rightarrow$ $S_\alpha = T^*_{s_\alpha} + T^*_{id}$ acts on $H_{top}(\mathbb{P}^1)$ by -1

$H_{top}(\mathbb{P}^1) = \text{sign rep}$

b) $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mu^{-1}(x) = pt \in T^*\mathbb{P}^1$

$[\mathbb{P}^1 \times \mathbb{P}^1] * [pt] = 0$

$\Rightarrow$ S_α acts by id on $H_{top}(\pi^{-1}(x))$.

$H_{top}(\pi^{-1}(0)) = \text{sign rep.}$

$H_{top}\left(\pi^{-1}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\right) = \text{trivial rep.}$