1) Lagrangian Construction of the Weyl group.

Recall

$$
\begin{tikzcd}
 & \tilde{N} \arrow[r, hook] \arrow[dl, "\mu"'] & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, two heads, "\nu"] & \\
N \arrow[r, hook] \arrow[dr, two heads] & \mathfrak{g} \arrow[dr, two heads] & & \mathfrak{h} \arrow[dl] \\
 & 0 \arrow[r, hook] & \mathfrak{h}/\!/W &
\end{tikzcd}
$$

$Z = \tilde{N} \times_N \tilde{N}$

<u>Thm</u>: $H_{top}(Z) \simeq \mathbb{Q}[W]$ as ass. algs.

preparations: recall $\mu: \tilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ is a principle W-bundle.

for any $h \in \mathfrak{h}$,

$$\tilde{\mathfrak{g}}^h := \nu^{-1}(h) \simeq G \times_B (h + \mathfrak{n}).$$

$\nu: \tilde{\mathfrak{g}}^{rs} \to \mathfrak{h}$ is W-equivariant.

Let h be a semisimple regular element.

$w \in W$, $\quad w: \tilde{\mathfrak{g}}^h \to \tilde{\mathfrak{g}}^{w(h)}$.

Let $\Lambda_w^h \subseteq \tilde{\mathfrak{g}}^{w(h)} \times \tilde{\mathfrak{g}}^h$ denote the graph of the w-action.

Hence $\Lambda_w^h = \left\{ (x, \mathfrak{b}, x', \mathfrak{b}') \;\middle|\; \begin{array}{c} x = x' \in \mathfrak{b} \cap \mathfrak{b}', \ \nu(x', \mathfrak{b}') = h, \\ (\mathfrak{b}', \mathfrak{b}) \in Y_w \end{array} \right\}$

$Y_w := G.(\mathfrak{b}, w(\mathfrak{b})) \subseteq \mathcal{B} \times \mathcal{B}$ orbit corr. to w.

<u>Lemma</u>:

$$\begin{tikzcd} \Lambda_w^h \arrow[r, "\sim"] \arrow[dr, "\pi^2"'] & G \times_{B \cap w(B)} (h + \mathfrak{n} \cap w(\mathfrak{n})) \arrow[d] \\ & Y_w \cong G/B \cap w(B) \end{tikzcd}$$

$$\tilde{\mathfrak{g}}^h \xrightarrow{w} \tilde{\mathfrak{g}}^{w(h)} \xrightarrow{y} \tilde{\mathfrak{g}}^{yw(h)}.$$

$\Rightarrow \Lambda_{yw}^h = \Lambda_y^{w(h)} \circ \Lambda_w^h$, and

$$[\Lambda_{yw}^h] = [\Lambda_y^{w(h)}] * [\Lambda_w^h].$$

proof of the theorem:

idea: Study the above via $\mathfrak{h} \to 0$, use specialization in Borel-Moore homology.

(Recall we need a local trivial fibration $Z(S^*) \to S^* = S \setminus \{0\}$)

We can't take the base S to be $\mathfrak{h}$ as the local trivial fibration condition will not be satisfied.

Instead, take a complex line $\mathfrak{l} \subseteq \mathfrak{h}^{reg}$, set $\mathfrak{l}^* = \mathfrak{l} \setminus \{0\}$.

Let $\tilde{\mathfrak{g}}^{\mathfrak{l}} := \nu^{-1}(\mathfrak{l})$, $\tilde{\mathfrak{g}}^{w(\mathfrak{l})} := \nu^{-1}(w(\mathfrak{l}))$

Define $$\Lambda_w^{\mathfrak{l}} := \left(\tilde{\mathfrak{g}}^{w(\mathfrak{l})} \times_{\mathrm{Graph}(\mathfrak{l} \xrightarrow{w} w(\mathfrak{l}))} \tilde{\mathfrak{g}}^{\mathfrak{l}}\right) \cap (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}).$$

Apply the specialization construction to

$$\nu: \Lambda_w^{\mathfrak{l}} \to \mathfrak{l}.$$

$$\Lambda_w^{0} = (\tilde{\mathcal{N}} \times \tilde{\mathcal{N}}) \cap (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}) = Z$$

Therefore, specialization defines

$m = \dim_{\mathbb{R}} Z$

$$\lim_{h \to 0} : H_{m+2}(\Lambda_w^{\iota^*}) \to H_m(Z), \quad h \text{ a varying pt in } L.$$

Let $[\Lambda_w^{0 \cdot h}] := \lim_{h \to 0} [\Lambda_w^{\iota^*}]$.

Fact: $[\Lambda_w^{0 \cdot h}]$ doesn't depend on h. Denote it by $[\Lambda_w^{\circ}]$.

($\Leftarrow$ transitivity of specialization).

On the other hand,

$$[\Lambda_{yw}^{yw(\iota^*)}] = [\Lambda_y^{w(\iota^*)}] * [\Lambda_w^{\iota^*}].$$

$$\xrightarrow[\text{Convolution.}]{\text{specialization commutes with}} [\Lambda_{yw}^{\circ}] = [\Lambda_y^{\circ}] * [\Lambda_w^{\circ}]$$

Define $\mathbb{Q}[W] \to H_{top}(Z)$

$$w \mapsto [\Lambda_w^{\circ}]$$

It remains to show that $\{[\Lambda_w^0] \mid w \in W\}$ is a basis for $H_{top}(Z)$

We already know $\{[\overline{T^*_{Y_w}(B\times B)}] \mid w \in W\}$ is a basis for $H_{top}(Z)$.

Notice that $\pi^2(\Lambda_w^{\mathfrak{l}^*}) \subseteq Y_w$

$$\Rightarrow \quad \Lambda_w^0 = \sum_{y \leq w} n_{w,y} \cdot [\overline{T^*_{Y_y}(B\times B)}], \quad n_{w,y} \in \mathbb{Q}.$$

$(\, y \leq w$ is $Y(y) \subseteq \overline{Y(w)}\,)$

Claim: $n_{w,w} = 1$.

Restrict to $Y(w)$, Λ_w^h, $h \in \mathfrak{l}^*$ is isomorphic to the flat family of affine bundles

$$G \times_{B \cap w(B)} (h + \mathfrak{n} \cap w(\mathfrak{n})) \longrightarrow Y_w.$$

which degenerates to

$$G \times_{B \cap w(B)} (\mathfrak{n} \cap w(\mathfrak{n})) \longrightarrow Y_w \quad \text{as } h \to 0.$$

This is exactly the conormal bundle $T^*_{Y_w}(\mathcal{B} \times \mathcal{B})$.

Thus, $n_{w,w} = 1$, and we are done. $\square$

Open question: what is $n_{w,y}$?

Side Remark

$$Z = \left\{ (g_1\mathfrak{b}, g_2\mathfrak{b}, x) \;\middle|\; \begin{array}{c} g_1, g_2 \in G, x \in \mathcal{N} \\ x \in g_1\mathfrak{n} \cap g_2\mathfrak{n} \end{array} \right\}$$

$$\begin{array}{ccc} \downarrow & & \overline{\mathfrak{b}} \\ \mathcal{B} & & g_1\mathfrak{b} \end{array}$$

the fiber over $\mathfrak{b}$ $= \left\{ (g_2\mathfrak{b}, x) \;\middle|\; \begin{array}{c} g_2 \in G, x \in \mathcal{N} \\ x \in g_2\mathfrak{n} \cap \mathfrak{n} \end{array} \right\}$

$$G \times_B \mathfrak{n} \simeq T^*\mathcal{B} \xhookrightarrow{\varphi} G/B \times \mathcal{N}$$

$$(g, \alpha) = (g.\mathfrak{b}, g\alpha g^{-1}) \longmapsto (g.\mathfrak{b}, g\alpha g^{-1})$$

$$\varphi^{-1}(G/B \times \mathfrak{n}) = \left\{ (g\mathfrak{b}, x) \;\middle|\; \begin{array}{c} g \in G, x \in \mathcal{N} \\ x \in \mathfrak{n} \cap g\mathfrak{n} \end{array} \right\} =: R$$

$$\Longrightarrow \quad Z \simeq G \times_B R.$$

Lusztig. $H_*^{G\times\mathbb{C}^*,BM}(Z) \simeq$ degenerate affine Hecke alg

$\quad\wr\!\shortparallel$

$$H_*^{G\times\mathbb{C}^*,BM}(G\times_B R) \simeq H_*^{B\times\mathbb{C}^*,BM}(R)$$

Recent work of Braverman-Finkelberg-Nakajima on Coulomb branch.

$(G, N\oplus N^*)$, N a finite dim'l rep of G

$\rightsquigarrow$ $\mathcal{M}_c(G,N) = \mathrm{Spec}(\text{some comm. alg } A)$

Coulomb branch.

How to construct this alg A ?

$K = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$,

$Gr := G(K)/G(\mathcal{O})$ affine Grassmannian. $\quad G/B$

$N_{\mathcal{O}} := N[[t]] \quad \quad \cap$

$N_K := N((t)) \quad \quad N$

$$T := G(K) \times_{G(\mathcal{O})} \mathcal{N}_{\mathcal{O}} \xrightarrow{\text{mult}} \mathcal{N}_K \qquad\qquad G \times_B \mathfrak{n} \longrightarrow \mathcal{N}$$

$$\downarrow \text{pr}_1 \qquad\qquad\qquad \downarrow$$

$$Gr = G(K)/G(\mathcal{O}) \qquad\qquad \mathcal{B}.$$

Define $R^{BFN} := (\text{pr}_1 \times \text{mult})^{-1}(Gr \times \mathcal{N}_{\mathcal{O}})$ $\qquad R$

Consider $H_*^{G(\mathcal{O}),BM}(R^{BFN})$ $\qquad H_*^{B,BM}(R)$

$$\mathrel{\substack{\cdot\cdot\\ \cong \\ \cdot\cdot}} \qquad\qquad\qquad \text{IS}$$

$$H_*^{G(K),BM}(T \times_{\mathcal{N}_K} T). \qquad\qquad H_*^{G,BM}(Z).$$

BFN constructed a convolution alg structure on $H_*^{G(\mathcal{O}),BM}(R^{BFN})$, and proved that it's commutative.

$$\overset{\cdot\cdot}{\parallel}$$

$$A$$

2) Geometric Analysis of $H(Z)$-action.

$\mu : \tilde{N} \to N$, $Z = \tilde{N} \times_N \tilde{N}$, $x \in N$, $\mathcal{B}_x := \mu^{-1}(x)$.

We already proved $H(Z) \simeq \mathbb{Q}[W]$.

We will obtain all irr. reps of W via $H(\mathcal{B}_x)$

Write $H(\mathcal{B}_x) := H_{top}(\mathcal{B}_x)$

Note $Z \circ \mathcal{B}_x = \mathcal{B}_x$, and $\mathcal{B}_x \circ Z = \mathcal{B}_x$.

$\Rightarrow$ $H(\mathcal{B}_x)$ is a $H(Z)$-bimodule.

$H(\mathcal{B}_x)_L / H(\mathcal{B}_x)_R$ left/right $H(Z)$-module.

$g \in G$, $\mathcal{B}_x \longrightarrow \mathcal{B}_{g(x)}$.

$g : H_*(\mathcal{B}_x) \longrightarrow H_*(\mathcal{B}_{g(x)})$, $c \mapsto g \cdot c$

<u>Lem</u>: the left (resp. right) $H(Z)$-action on $H_*(\mathcal{B}_x)$ is compatible with the natural G-action. i.e. $g.(z \cdot c) = z \cdot (g.c)$

for $g \in G$, $z \in H(Z)$, $c \in H_*(\mathcal{B}_x)$.

<u>Pf:</u> G also acts on Z, $g: \mathcal{B}_x \to \mathcal{B}_{g(x)}$

$$g.(z.c) = g(z)\cdot g(c)$$

But G is connected $\Rightarrow$ G action on $H(Z)$ is trivial. $\square$

$G_x = Z_G(x) \curvearrowright \mathcal{B}_x$ by conjugation

$\Rightarrow$ $C(x) := G_x / G_x^\circ$ the conn. component group acts

on $H(\mathcal{B}_x)$.

<u>Lem:</u> $\exists$ natural $C(x)$-action on $H(\mathcal{B}_x)$, which commutes with the $H(Z)$-action.

Therefore, we have decomposition:

$$\mathbb{C} \otimes_{\mathbb{Q}} H(\mathcal{B}_x)_L = \bigoplus_{\chi \in C(x)^\wedge} \chi \otimes H(\mathcal{B}_x)_\chi \quad \cdots\cdots (*)$$

$C(x)^\wedge = \{\text{complex irreps of } C(x) \text{ which occur in } H(\mathcal{B}_x)\} / \simeq$

Remark: We need to use $\mathbb{C}$-coefficient since not all irreps of $C(x)$ can be defined over $\mathbb{Q}$.

Thm: a) $\forall x \in \mathcal{N}$, $\chi \in C(x)^\wedge$, $H(\mathcal{B}_x)_\chi$ is a simple $H(\mathfrak{Z})$-mod.

b). $H(\mathcal{B}_x)_\chi$ and $H(\mathcal{B}_y)_\psi$ are isomorphic iff the pairs (x, χ) and (y, ψ) are G-conjugate.

c). The set $\{ H(\mathcal{B}_x)_\chi \mid x \in \mathcal{N}, \chi \in C(x)^\wedge \}$ is a complete collection of isomorphism classes of simple $H(\mathfrak{Z})$-mods.

Rmk: The proof of thm holds in a more general setting:

- $\tilde{\mathcal{N}}$ smooth G-variety
- $\mathcal{N}$ G-variety consisting of finitely many G-orbits
- $\mu: \tilde{\mathcal{N}} \to \mathcal{N}$ G-equivariant, semi-small.

We will work in the Springer resolution case.

To prove this, we need some preparations.

Let $(H(\mathbb{B}_x)_L)^\vee := \mathrm{Hom}_{\mathbb{Q}}(H(\mathbb{B}_x)_L, \mathbb{Q})$, and define a right $H(2)$-action on it by

$$(\check{v}\cdot z)(w) = \check{v}(z\cdot w), \qquad z\in H(2),\ \check{v}\in (H(\mathbb{B}_x)_L)^\vee,$$
$$w\in H(\mathbb{B}_x)_L.$$

Lemma 1: $\exists$ an isomorphism of right $H(2)$-modules

$$H(\mathbb{B}_x)_R \simeq (H(\mathbb{B}_x)_L)^\vee,$$ which is compatible with the $C(x)$-actions, where $C(x)$ acts on $(H(\mathbb{B}_x)_L)^\vee$ by

$$(g\cdot\check{v})(w) = \check{v}(g^{-1}w), \quad \check{v}\in (H(\mathbb{B}_x)_L)^\vee,\ g\in C(x),\ w\in H(\mathbb{B}_x)_L.$$

Lemma 2: $H(2)$ is a semisimple alg.

Rmk: Lem 1 & 2 can be proved use sheaf-theoretic techniques in the more general setting. Lemma 2 already holds in the

Springer case as $H(Z) \cong \mathbb{Q}[W]$

pf of the theorem based on Lemma 1 and 2:

Introduce a partial order on the set of nilpotent orbits $\mathcal{O} \subseteq \mathcal{N}$.

$\mathcal{O}' \leq \mathcal{O}$ if $\mathcal{O}' \subseteq \overline{\mathcal{O}}$.

$\mathcal{O}' < \mathcal{O}$ if $\mathcal{O}' \subseteq \overline{\mathcal{O}} \setminus \mathcal{O}$.

$\mu_Z : Z \to \mathcal{N}$, for any $S \subseteq \mathcal{N}$, Let $Z_S := \mu_Z^{-1}(S)$

Let $Z_{\leq \mathcal{O}} := \bigsqcup_{\mathcal{O}' \leq \mathcal{O}} Z_{\mathcal{O}} = Z_{\overline{\mathcal{O}}}$,

$Z_{<\mathcal{O}} := \bigsqcup_{\mathcal{O}' < \mathcal{O}} Z_{\mathcal{O}'}$.

Then $Z \circ Z_{\leq \mathcal{O}} = Z_{\leq \mathcal{O}} \circ Z = Z_{\leq \mathcal{O}}$.

<u>Cor</u>: $H(Z_{<\mathcal{O}})$ and $H(Z_{\leq \mathcal{O}})$ are 2-sided ideals in $H(Z)$.

put $H_{\mathcal{O}} := H(Z_{\leq \mathcal{O}}) / H(Z_{<\mathcal{O}})$.

$H_{\mathcal{O}}$ is a $H(Z)$ bi-mod, and it has a basis formed by the fundamental classes of the irr. comps of $Z_{\mathcal{O}}$.

Recall $Z_{\mathcal{O}} = \mu^{-1}(\mathcal{O}) = \widetilde{\mathcal{O}} \times_{\mathcal{O}} \widetilde{\mathcal{O}}$

$$= G \times_{G_x} (\mathcal{B}_x \times \mathcal{B}_x)$$

irr. comps of $Z_{\mathcal{O}}$ are of the form

$$G \times_{G_x} (\mathcal{B}_x^{\alpha} \times \mathcal{B}_x^{\beta}), \quad \text{where } \{\mathcal{B}_x^{\alpha}\} = \text{irr. comps of } \mathcal{B}_x.$$

Hence, $\{\text{irr. comps of } Z_{\mathcal{O}}\} \xleftrightarrow{\text{bijection}}$ $C(x)$-orbits on pairs of comps of $\mathcal{B}_x$.

The natural restriction map (Lemma 2.7.46) gives an alg homomorphism

$$H_{\mathcal{O}} \to H(Z_x) = H(\mathcal{B}_x \times \mathcal{B}_x) = H(\mathcal{B}_x) \times H(\mathcal{B}_x)$$

Moreover, by the description of the basis of $H_{\mathcal{O}}$, we get an $H(Z)$-bimod isomorphism.

$$\boxed{H_{\mathcal{O}} \xrightarrow{\sim} \left(H(\mathcal{B}_x)_L \otimes H(\mathcal{B}_x)_R\right)^{C(x)}}$$

The closure relation $\leq$ on the orbits $\mathcal{O}$ gives a filtration of $H(Z)$ by the two sided ideals $H(Z_{\leq \mathcal{O}})$.

$\mathrm{gr}\,H(Z)$ = the ass. graded space w.r.t. to this filtration, which is a $H(Z)$-bimod.

Moreover, $H(Z) \simeq \mathrm{gr}H(Z)$ as $H(Z)$-bimod since $H(Z)$ is a semisimple alg. (Lemma 2).

Thus

$$H(Z) \simeq \mathrm{gr}H(Z) = \bigoplus_{\mathcal{O} \subseteq N} H_{\mathcal{O}}$$

$$= \bigoplus_{\mathcal{O} \subseteq N} \left(H(\mathcal{B}_x)_L \otimes H(\mathcal{B}_x)_R\right)^{C(x)}$$

$$(\text{Lemma } 1) \quad = \bigoplus_{\mathcal{O} \subseteq N} \mathrm{Hom}_{C(x)}\left(H(\mathcal{B}_x)_L, H(\mathcal{B}_x)_L\right)$$

$$\Rightarrow \quad \mathbb{C}\otimes_{\mathbb{Q}} H(2) \overset{\text{equation}(*)}{=} \bigoplus_{\left(\substack{x\in N \\ \chi,\psi\in C(x)^\wedge}\right)/\mathrm{conj}} \mathrm{Hom}_{C(x)}(\chi,\psi)\otimes \mathrm{Hom}_{\mathbb{C}}(H(\mathcal{B}_x)_\chi, H(\mathcal{B}_x)_\psi)$$

$$= \bigoplus_{\left(\substack{x\in N \\ \chi\in C(x)^\wedge}\right)/\mathrm{conj}} \mathrm{Hom}_{\mathbb{C}}(H(\mathcal{B}_x)_\chi, H(\mathcal{B}_x)_\chi). \quad \text{—} \quad (**)$$

as $H(2)$-bimods.

On the other hand, let $\{E_\alpha\}$ be a complete collection of simple $H(2)$-mods, since $\mathbb{C}\otimes_{\mathbb{Q}} H(2)$ is semisimple,

$$\mathbb{C}\otimes_{\mathbb{Q}} H(2) \simeq \bigoplus_{\alpha} \mathrm{Hom}_{\mathbb{C}}(E_\alpha, E_\alpha) \quad \text{—} \quad (***)$$

compare $(**)$ and $(***)$, we see each

$H(\mathcal{B}_x)_\chi$ must be a simple $H(2)$-mod, and

$\{H(\mathcal{B}_x)_\chi \mid x\in N, \chi\in C(x)^\wedge\}$ is a complete collection of isomorphism classes of simple $H(2)$-mods. $\square$