1) Continuation of the Springer theory.

Now let's prove lemma 1 in the Springer resolution case.

$\exists$ isomorphism of right $H(Z)$-mods.

$$H(\mathcal{B}_x)_R \simeq (H(\mathcal{B}_x)_L)^\vee$$

pf:

$$Z \subseteq \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$$

Switching the factors gives an involution on $Z$.

$\rightsquigarrow$ $c \mapsto c^t$ on $H(Z)$

It's easy to see $(c_1 * c_2)^t = c_2^t * c_1^t$.

Hence, $c \mapsto c^t$ is an alg. involution on $H(Z)$.

<u>Lemma</u> Under the isomorphism $H(Z) \simeq \mathbb{Q}[W]$,

$c \mapsto c^t$ corresponds to $w \mapsto w^{-1}$ on $\mathbb{Q}[W]$.

pf: for a regular $h \in \mathfrak{h}$,

$\Lambda_w^h$ = graph of $(\tilde{\mathfrak{g}}^h \xrightarrow{\sim} \mathfrak{g}^{w(h)})$

Then $(\Lambda_w^h)^t = \Lambda_{w^{-1}}^{w(h)}$

Hence $(\Lambda_w^0)^t = \Lambda_{w^{-1}}^0$. $\square$

Define a left $H(2)$-mod structure on $H(\mathcal{B}_x)_R$ by

$c \cdot v := v \cdot c^t$, where $c \in H(2)$, $v \in H(\mathcal{B}_x)_R$.

and denote this left module by $(H(\mathcal{B}_x)_R)^t$.

Then we have

$$(H(\mathcal{B}_x)_R)^t \simeq H(\mathcal{B}_x)_L.$$

Therefore, we need to show

$$((H(\mathcal{B}_x)_L)^\vee)^t \simeq H(\mathcal{B}_x)_L$$

Using the fact that $c \mapsto c^t$ corresponds to $w \mapsto w^{-1}$,

$((H(\mathcal{B}_x)_L)^\vee)^t$ = contragredient $\mathbb{Q}[W]$-module.

Hence, it's isomorphic to $H(\mathcal{B}_x)_L$, since it admits a

$W$-inv. non-degenerate bilinear form $\square$

This finishes the proof of the main theorem.

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Thm (Springer classification of simple $W$-modules)

The set $\{ H(\mathcal{B}_x)_\psi \mid G$-conjugacy classes of pairs $(x \in \mathcal{N}, \psi \in c(x)^\wedge) \}$ is the complete list of isomorphism classes of simple $W$-modules.

<u>Ex</u>: 1) $x \in \mathcal{N}^{reg}$, $\mathcal{B}_x = 1$ point, $H(\mathcal{B}_x) =$ trivial rep.

2) $x = 0$, $\mathcal{B}_x = \mathcal{B}$, $H(\mathcal{B}) = \mathbb{Q}[[\mathcal{B}]] =$ sign rep.

Type A case.

Thm (Irreps of $S_n$).

$G = SL_n(\mathbb{C})$,

$\{H(\mathcal{B}_x) \mid x \in \mathcal{O} \subseteq \mathcal{N}\}$ is a complete collection of isomorphism

classes of simple $H(\mathcal{Z}) = \mathbb{Q}[S_n]$-mods.

Pf: Replace $G = SL_n(\mathbb{C})$ by $GL_n(\mathbb{C})$,

$\mathcal{B}$, $\mathcal{N}$, $\mathcal{Z}$, $W$ are unchanged.

Lemma Let $G = GL_n(\mathbb{C})$, $\forall\ x \in \mathfrak{g}$, $G_x = Z_G(x)$ is

connected. So that $C(x) = 1$.

Pf: $G_x = \{ y \in M_n(\mathbb{C}) \mid xy = yx,\ \det y \neq 0 \}$.

$xy = yx$ defines a vector subspace $V$ in $M_n(\mathbb{C})$.

$\det y \neq 0$ gives a complement of a complex hypersurface in $V$.

which is of real codim 2. Hence connected. $\square$

$\square$

Rmk: 1) Irreps of $S_n \xleftrightarrow{1:1}$ Partitions of $n$.

2)

$$\# S_n = \sum_{V \in \hat{S}_n} (\dim V)^2$$

$Z = \bigsqcup_w T^*_{Y_w}(\mathcal{B}\times\mathcal{B}) \to$ ||

$$\# \text{Irr. comps of } Z$$

$Z = \bigsqcup_{x \in \mathcal{O} \subseteq \mathcal{N}} G \times_{G_x} (\mathcal{B}^\alpha_x \times \mathcal{B}^\beta_x)$ ||

$$\sum_{x \in \mathcal{O} \subseteq \mathcal{N}} (\#\text{ irr. comps of } \mathcal{B}_x)^2 = \sum_{x \in \mathcal{O} \subseteq \mathcal{N}} (\dim H(\mathcal{B}_x))^2.$$

$\dim H(\mathcal{B}_x) = \#$ irr. comps of $\mathcal{B}_x$

3) $\#$ involutions in $S_n = \sum_{V \in \hat{S}_n} \dim V$

$Z = \bigsqcup_w T^*_{Y_w}(\mathcal{B}\times\mathcal{B})$ ||

|| $\dim H(\mathcal{B}_x) = \#$ irr. comps of $\mathcal{B}_x$

$\#$ irr. comps of $Z$ fixed under switching the factors $= \sum_{x \in \mathcal{O} \subseteq \mathcal{N}} \#$ irr. comps of $\mathcal{B}_x$

$Z = \bigsqcup_{\mathcal{O}} G \times_{G_x} (\mathcal{B}^\alpha_x \times \mathcal{B}^\beta_x)$

The Weyl group action on $H^*(\mathcal{B}, \mathbb{Q})$.

$x = 0$, $\mathcal{B}_x = \mathcal{B}$

$$G/T \xrightarrow{p} G/B = \mathcal{B}. \qquad \text{fibers are contractible}$$

$$\rightsquigarrow p_* : H_*^{ord}(G/T) \xrightarrow{\sim} H_*^{ord}(\mathcal{B}) = H_*(\mathcal{B}).$$

$W$ acts on $G/T$ $\Rightarrow$ $W \circlearrowright H_*(\mathcal{B})$.

<u>Lemma</u>: This action = the action from convolution.

pf: $h \in \mathfrak{h}^{reg}$.

$$\tilde{\mathfrak{g}}^h := \nu^{-1}(h) = G \times_B (h + \mathfrak{n})$$

$$\begin{tikzcd}
 & \tilde{\mathfrak{g}}^h \arrow[dl, "\mu"'] \arrow[dr, "\tilde{\pi}"] & \\
\mathrm{Ad}\,G.h \simeq G/T \arrow[rr, "p"] & & G/B
\end{tikzcd}$$

$\mu$ is an isomorphism. ($\Leftarrow$ $h + \mathfrak{n} = B.h$ Lemma 3.1.44)

$$
\begin{tikzcd}
 & H_*^{ord}(\tilde{\mathcal{Y}}^h) \arrow[dl, equal, "\mu_*"'] \arrow[dr, equal, "\pi_*"] & \\
H_*^{ord}(G/T) \arrow[rr, equal, "p_*"] & & H_*(\mathcal{B})
\end{tikzcd}
$$

Further,

$$
\begin{tikzcd}
\tilde{\mathcal{Y}}^h \arrow[r, "w\text{-action}"] \arrow[d, equal, "\mu"] \arrow[dr, phantom, "G \circlearrowright"] & \tilde{\mathcal{Y}}^{w(h)} \arrow[d, equal, "\mu"] \\
G/T \arrow[r, "w\text{-action}"] & G/T
\end{tikzcd}
$$

$\rightsquigarrow$

$$
\begin{tikzcd}
H_*(\mathcal{B}) \arrow[rrr, dashed, "\text{conv with } {[\Lambda_w^h]}"] \arrow[ddd, equal] \arrow[dr, equal] & & & H_*(\mathcal{B}) \\
 & H_*^{ord}(\tilde{\mathcal{Y}}^h) \arrow[r, "\text{conv. with}", "{[\Lambda_w^h]}"'] \arrow[ddl, equal, "\mu_*"'] & H_*^{ord}(\tilde{\mathcal{Y}}^{wh}) \arrow[ur, equal] \arrow[ddr, equal] & \\
 & & & \\
H_*^{ord}(G/T) \arrow[rrr, "w\text{-action}"] & & & H_*^{ord}(G/T)
\end{tikzcd}
$$

$\Rightarrow$ take specialization. (which commutes with convolution)

$$
\begin{tikzcd}
H_*(\mathcal{B}) \arrow[rrr, "\text{conv with } [\Lambda_w^\circ]"] \arrow[dr, equal] \arrow[dd, equal] & & & H_*(\mathcal{B}) \arrow[dd, equal] \\
& H_*^{ord}(T^*\mathcal{B}) \arrow[r, "\text{conv.}"', "{[\Lambda_w^\circ]}"' near end] & H_*^{ord}(T^*\mathcal{B}) \arrow[ur, equal] \arrow[dr, equal] & \\
H_*^{ord}(G/T) \arrow[ur, equal, "\gamma"'] \arrow[rrr, "w\text{-action}"] & & & H_*^{ord}(G/T)
\end{tikzcd}
$$

$\gamma$ = specialization of $\mu_*$.

(induced by a diffeomorphism $G/T \simeq T^*\mathcal{B}$)

$\square$

2) Constructible Sheaves.

$X$ a reasonable topological space. (locally compact, $X$ admit a closed embedding into a $C^\infty$-manifold, $\exists$ open nbd $U \supseteq X$ in $M$, s.t. $X$ is a homotopy retract of $U$).

$Sh(X)$ = abelian category of $\mathbb{C}$-vector spaces on $X$.

Ex: 1) $x \in X$, $V$ is a $\mathbb{C}$-vector space.

sky-scraper sheaf $V_x$,

$$V_x(U) = \begin{cases} V & x \in U \\ 0 & \text{otherwise.} \end{cases}$$

2) constant sheaf $V_X$

$V_X(U) = V$ for any conn. open subset $U$.

3) locally constant sheaf. (local systems)

sheaf $\mathcal{F}$, $\forall x \in X$, $\exists$ an open nbd. $U$, st. $\mathcal{F}|_U$ is constant. (finite dim'l stalks)

If $X$ is connected, then

$$\left\{ \begin{matrix} \text{local systems} \\ \text{on } X \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{finite dim'l} \\ \pi_1(X, x)\text{-mods} \end{matrix} \right\}$$

by taking the monodromy rep. at $x$.

Recall, $\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$.

---

$f: X \to Y$ continuous

a) pullback functor $f^*: Sh(Y) \to Sh(X)$

$$\begin{matrix} Sh(Y) \\ \cup \\ \mathcal{G}, \end{matrix}$$

$f^*(\mathcal{G})$ is the sheafification of the presheaf

$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) \qquad U \subseteq X, \ V \subseteq Y \text{ open}$$

then $({}^\circ f \circ {}^\circ g)^* = {}^\circ g^* \circ {}^\circ f^*$, and $({}^\circ f^* \mathcal{F})_x = \mathcal{F}_{f(x)}$.

b). pushforward. ${}^\circ f_*: Sh(X) \to Sh(Y)$

$$\begin{array}{c} Sh(X) \\ \cup \\ \mathcal{F} \end{array}$$

$({}^\circ f_* \mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$.

$({}^\circ f \circ {}^\circ g)_* = {}^\circ f_* \circ {}^\circ g_*$

$Y = pt$, ${}^\circ f_* = \Gamma(X, -)$ global sections functor.

c). Shriek push-forward. $f_!$ (relative version of $\Gamma_c$ = global sections with compact support).

$({}^\circ f_! \mathcal{F})(V) = \{ s \in \mathcal{F}(f^{-1}(V)) \mid f|_{supp(s)} : Supp(s) \to V \text{ is proper} \}$

Hence, $\exists$ natural inclusion ${}^\circ f_! \mathcal{F} \hookrightarrow {}^\circ f_* \mathcal{F}$.

if $f$ is proper, ${}^\circ f_! = {}^\circ f_*$.

Moreover, if $f: X \hookrightarrow Y$ is a locally closed inclusion,

$f_! =$ extension by zero functor.

d). Internal $\mathcal{H}om$.

$\mathcal{F}_1, \mathcal{F}_2 \in Sh(X)$.

$$\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)(U) := \mathrm{Hom}_{Sh(U)}(\mathcal{F}_1|_U, \mathcal{F}_2|_U)$$

---

$X$ complex alg variety.

<u>Def</u>. a finite partition $X = \sqcup_i X_i$ is called a stratification of $X$ if:

a) Each $X_i$ is a smooth locally closed alg. subvariety of $X$

b) $\overline{X_j}$ is a union of the $X_i$'s.

<u>Examples</u>: a) $X = G/B = \bigsqcup_{w \in W} BwB/B$

b) the nilpotent cone $\mathcal{N} = \coprod \mathcal{O}$, $\mathcal{O}$ is a G-orbit

---

<u>Def</u>: a) $\mathcal{F} \in Sh(X)$ is called constructible if $\exists$ a stratification $X = \coprod X_\alpha$, s.t. $\forall \alpha$, $\mathcal{F}|_{X_\alpha}$ is a locally constant sheaf of finite dim'l vector spaces.

b) an object $A^\cdot \in D^b(Sh(X))$ is called a constructible complex if all the cohomology sheaves

$$\mathcal{H}^i(A^\cdot) := \ker(A^i \to A^{i+1}) / \operatorname{Im}(A^{i-1} \to A^i)$$

are constructible.

$D^b_c(X) :=$ full subcat of $D^b(Sh(X))$ formed by constructible complexes

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