1) Constructible sheaf continued.

dualizing complex

$i: X \hookrightarrow M$ closed embedding, M smooth.

define restriction with supports functor.

$$ {}^\circ i^! : Sh(M) \to Sh(X). $$

${}^\circ i^! \mathcal{G}$ is the sheafification of the following presheaf.

$$ {}^\circ i^! \mathcal{G}(U) := \varinjlim_{V,\, V \cap X = U} \{ s \in \mathcal{G}(V) \mid \mathrm{Supp}(s) \subseteq U \} $$

Let $\Gamma_{[X]}(V, \mathcal{G}) = \{ s \in \mathcal{G}(V) \mid \mathrm{supp}(s) \subseteq X \cap V \}$.

then $\forall x \in X$,

$$ ({}^\circ i^! \mathcal{G})_x = \varinjlim_{x \in V \subseteq M} \Gamma_{[X]}(V, \mathcal{G}). \qquad \text{Let } i^! = R\,{}^\circ i^! $$

Define: The dualizing complex of X is

$$ \mathbb{D}_X := i^! (\mathbb{C}_M) [2 \dim_{\mathbb{C}} M]. $$

Rmks: 1) $\mathcal{H}^j_x(\mathbb{D}_X) = H^{j+2\dim_{\mathbb{C}} M}(U, U\setminus U\cap X) \overset{\text{Poincare}}{\simeq} H_{-j}(U\cap X)$, $x\in X$

$U\subseteq M$ is a small contractible open nbd. of x in M.

2) $\mathbb{D}_X$ doesn't depend on the choice of $X \hookrightarrow M$.

3). if X is smooth, then $\mathbb{D}_X = \mathbb{C}_X[2\dim_{\mathbb{C}} X]$

Hypercohomology

$\mathcal{F}\in D^b_c(X)$, the hypercohomology group

$\mathbb{H}^i(\mathcal{F}) :=$ i-th derived functor to the global sections

function Γ.

Since $\Gamma(X,\mathcal{F}) \simeq \mathrm{Hom}_{sh(X)}(\mathbb{C}_X, \mathcal{F})$,

$$\mathbb{H}^i(X,\mathcal{F}) = R^i\Gamma(X,\mathcal{F}) = R^i\mathrm{Hom}(\mathbb{C}_X,\mathcal{F}) = \mathrm{Ext}^i_{D^b_c(X)}(\mathbb{C}_X,\mathcal{F})$$

To compute this, find a complex of injective sheaves

$I^\cdot$, which is quasi-isomorphic to $\mathcal{F}$,

$$\mathbb{H}^i(X,\mathcal{F}) := H^i(\Gamma(I^\cdot)) = H^i(\mathrm{Hom}_{sh(X)}(\mathbb{C}_X, I^\cdot))$$

We'll consider derived functors below.

$$H^i(X) = H^i(X, \mathbb{C}_X)$$

sheaf-theoretic definition of the Borel-Moore homology

$$H_i(X) = H^{-i}(X, \mathbb{D}_X)$$

($\mathbb{D}_X = i^! \mathbb{C}_M[2\dim_{\mathbb{C}} M]$, $H^{-i}(\mathbb{D}_X) = (2\dim_{\mathbb{C}} M - i)$-th hypercohomology

of $R\Gamma_{[X]}$ applied to the constant sheaf $\mathbb{C}_M$

$= H^{2\dim_{\mathbb{C}} M - i}(M, M \setminus M \cap X) \underset{\text{Poincare}}{\simeq} H_i(X)$)

If $Y \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U := X \setminus Y$, i closed embedding

$\forall \mathcal{F} \in D^b_c(X)$ $\quad i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^! \mathcal{F} \xrightarrow{+1}$

$$\Rightarrow \cdots \to H^k(Y, i^! \mathcal{F}) \to H^k(X, \mathcal{F}) \to H^k(U, j^* \mathcal{F}) \to H^{k+1}(Y, i^! \mathcal{F}) \to \cdots$$

Let $\mathcal{F} = \mathbb{D}_X$, we get the long exact sequence in Borel-Moore homology.

$A, B \in D^b(X)$.

define $\mathrm{Ext}^k_{D^b_c(X)}(A, B) := \mathrm{Hom}_{D^b_c(X)}(A, B[k])$

$$\mathrm{Ext}^{\bullet}_{D^b_c(X)}(A, B) = H^{\bullet}(X, \mathcal{H}om(A, B))$$

Verdier dual

$$A \in D^b_c(X) \longrightarrow \mathcal{H}om(A, \mathbb{D}_X) =: A^{\vee}.$$
$$=: D_X(A)$$

Hence, $\mathbb{C}_X^{\vee} = \mathbb{D}_X$

$$(A[n])^{\vee} = A^{\vee}[-n] \quad , \quad (A^{\vee})^{\vee} = A.$$

$f: X_1 \to X_2$,

Let $f_*: D_c^b(X_1) \to D_c^b(X_2)$, $f^*: D_c^b(X_2) \to D_c^b(X_1)$

denote the corresponding <u>derived functors</u>.

(1) also exists $f_!: D_c^b(X_1) \to D_c^b(X_2)$, $f^!: D_c^b(X_2) \to D_c^b(X_1)$, s.t.

$$f_! A_1 = (f_*(A_1^\vee))^\vee, \qquad f^! A_2 = (f^*(A_2^\vee))^\vee.$$

$$A_1 \in D_c^b(X_1), \quad A_2 \in D_c^b(X_2).$$

(2) $\mathrm{Hom}(f^*A_2, A_1) = \mathrm{Hom}(A_2, f_*A_1)$

$\mathrm{Hom}(A_1, f^! A_2) = \mathrm{Hom}(f_! A_1, A_2)$

(3) $f_! = f_*$ if f is proper

$f^! = f^*[2d]$ if f is smooth of relative dim d.

(4) $H^*(X_2, f_* A_1) = H^*(X_1, A_1)$, $H_c^*(X_2, f_! A_1) = H_c^*(X_1, A_1)$

$f^* \mathbb{C}_{X_2} = \mathbb{C}_{X_1}$, $f^! \mathbb{D}_{X_2} = \mathbb{D}_{X_1}$

Ex: $X_2 = pt$, X_1 smooth of dim d, $A_2 = \mathbb{C}$, $A_1 = \mathcal{L}$ a local system on X_1,

then $f^! A_2 = \mathbb{D}_{X_1}$,

$$\mathrm{Hom}_{X_1}(A_1, f^! A_2) = \mathrm{Hom}_{pt}(f_! A_1, A_2) = \mathrm{Hom}_{pt}(f_! \mathcal{L}, \mathbb{C}) = R\Gamma_c(\mathcal{L})^*$$

$$\begin{array}{l} \| \\ \mathrm{Hom}_{X_1}(A_1, f^* \mathbb{C}[2d]) = \mathrm{Hom}_{X_1}(\mathcal{L}, \mathbb{C}_{X_1}[2d]) = R\Gamma(\mathcal{L}^*[2d]) \end{array}$$

dual local system.

$$\Rightarrow H^j_c(X_1, \mathcal{L})^* \simeq H^{2d-j}(X_1, \mathcal{L}^*)$$

Poincare duality for local systems.

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base change formula.

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

$A \in D^b_c(X)$,

$$g^! f_* A \simeq \tilde{f}_* \tilde{g}^! A.$$

tensor products.

$i_\Delta : X \hookrightarrow X \times X$ diagonal embedding.

$A \otimes B := i_\Delta^* (A \boxtimes B)$, $\quad A \overset{!}{\otimes} B = i_\Delta^! (A \boxtimes B)$.

then $A \otimes \mathbb{C}_X = A$, $\quad A \overset{!}{\otimes} \mathbb{D}_X = A$, $\quad \mathcal{H}om(A,B) \simeq A^\vee \overset{!}{\otimes} B$.

$$\left(\begin{tikzcd} X \arrow[r, hook, "i_\Delta"] \arrow[dr, "id"'] & X \times X \arrow[d, "id \times \pi"] \\ & X \times pt \end{tikzcd} \qquad \begin{array}{l} \pi^* \mathbb{C}_{pt} = \mathbb{C}_X \\ \pi^! \mathbb{D}_{pt} = \mathbb{D}_X . \end{array} \right)$$

Take $A = B$, then

$$\mathrm{Ext}^\cdot_{D^b_c(X)}(A,A) = H^\cdot(\mathcal{H}om(A,A)) = H^\cdot(X, A^\vee \overset{!}{\otimes} A) = \mathrm{Ext}^\cdot(\mathbb{C}_X, A^\vee \overset{!}{\otimes} A).$$

Hence $id \in \mathrm{Ext}^0_{D^b_c(X)}(A,A)$ gives

$$\mathbb{C}_X \to A^\vee \overset{!}{\otimes} A .$$

By verdier duality, we get $A^\vee \otimes A \to \mathbb{D}_X$.

Revisit the Borel Moore homology

Recall $\mathrm{Hom}(f^*A_2, A_1) = \mathrm{Hom}(A_2, f_*A_1)$

Take $A_1 = f^*A_2$, we get a canonical morphism.

$$A_2 \to f_* f^* A_2.$$

Similarly, take $A_1 = f^! A_2$ in

$$\mathrm{Hom}(A_1, f^! A_2) = \mathrm{Hom}(f_! A_1, A_2)$$

we get a canonical morphism $f_! f^! A_2 \to A_2$.

Recall $H_i(X) := H^{-i}(X, \mathbb{D}_X) \simeq (H_c^i(X))^*$

$\mathrm{Hom}(\mathbb{C}_X, \omega_X) \simeq \mathrm{Hom}(p_{X!}\mathbb{C}_X, \mathbb{C}_{pt})$

1) proper pushforward.

$f: X_1 \to X_2$ proper, $f_* = f_!$

$\to f_* f^! \mathbb{D}_{X_2} = f_! f^! \mathbb{D}_{X_2} \to \mathbb{D}_{X_2}$

$\Rightarrow H^\cdot(X_1, \mathbb{D}_{X_1}) = H^\cdot(P_{1*}\mathbb{D}_{X_1}) = H^\cdot(P_{2*} f_* \mathbb{D}_{X_1})$

$$= H^\cdot(p_{2*} f_* f^! \mathbb{D}_{X_2}) \to H^\cdot(p_{2*}\mathbb{D}_{X_2}) = H^\cdot(X_2, \mathbb{D}_{X_2}).$$

where $p_i : X_i \to pt$ is the constant map.

2) restriction with supports

Consider
$$\begin{tikzcd} Y\cap Z \arrow[r, hook, "\tilde{i}"] \arrow[d, hook, "\tilde{j}"'] & Z \arrow[d, hook, "j"] \\ Y \arrow[r, hook, "i"] & X \end{tikzcd}$$

for $A \in D^b(X)$, $\exists\ A \to i_* i^* A$.

Hence $$j^! A \to j^! i_* i^* A \underbrace{=}_{\text{base change}} \tilde{i}_* \tilde{j}^! i^* A$$

Assume X is smooth, $i : Y \hookrightarrow X$ closed embedding of a complex codim d smooth subvariety.

$A = \mathbb{D}_X = \mathbb{C}_X[2\dim_{\mathbb{C}} X]$, then $i^* A = \mathbb{C}_Y[2\dim_{\mathbb{C}} Y + 2d] = \mathbb{D}_Y[2d]$

$j^! A = \mathbb{D}_Z$.

$\rightsquigarrow \mathbb{D}_Z \to \tilde{i}_* \tilde{j}^! \mathbb{D}_Y[2d] = \tilde{i}_* \mathbb{D}_{Y\cap Z}[2d]$

$\rightsquigarrow H_i(Z) = H^{-i}(Z, \mathbb{D}_Z) \to H^{-i+2d}(Z, \hat{i}_* \mathbb{D}_{Y\cap Z}) = H^{-i+2d}(Y\cap Z, \mathbb{D}_{Y\cap Z})$

$= H_{i-2d}(Y\cap Z)$

3) Smooth pullback.

$p: \tilde{X} \to X$ locally trivial oriented fibration with smooth fiber of real dim d.

then $p^! = p^*[d]$. (locally on $\tilde{X}$, $p = pr_1: X\times F \to X$)

$\forall A \in D_c^b(X)$, $\quad A \to p_* p^* A = p_* p^! A[-d]$.

Take $A = \mathbb{D}_X$, $\quad \mathbb{D}_X \to p_* \mathbb{D}_{\tilde{X}}[-d]$,

$\rightsquigarrow H_i(X) = H^{-i}(X, \mathbb{D}_X) \to H^{-i}(X, p_* \mathbb{D}_{\tilde{X}}[-d])$

$= H^{-i-d}(\tilde{X}, \mathbb{D}_{\tilde{X}}) = H_{i+d}(\tilde{X})$.

this is the smooth pullback p^*.

Let $i: X \to \tilde{X}$ be a continuous section of p.

Since locally on $\tilde{X}$, $i = id_X \times i_s : X \hookrightarrow X \times F$, for some point $s \in F$.

$$i^* \mathbb{D}_{\tilde{X}} = (id_X \times i_s)^* (\mathbb{D}_X \boxtimes \mathbb{D}_F) = \mathbb{D}_X \boxtimes \mathbb{C}_s[d] \simeq \mathbb{D}_X[d]$$

$$\Rightarrow \mathbb{D}_{\tilde{X}} \to i_* i^* \mathbb{D}_{\tilde{X}} = i_* \mathbb{D}_X[d].$$

$$\Rightarrow H_i(\tilde{X}) = H^{-i}(\tilde{X}, \mathbb{D}_{\tilde{X}}) \to H^{-i}(\tilde{X}, i_* \mathbb{D}_X[d])$$
$$= H^{d-i}(X, \mathbb{D}_X) = H_{i-d}(X).$$

this is the Gysin pull-back i^*.

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4) smooth base change.

Consider the Cartesian square

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{\phi}"] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "f"] \\ \tilde{X} \arrow[r, "\phi"] & X \end{tikzcd}$$

f proper,

ϕ locally trivial oriented fibration with smooth fiber of real dim d.

Then we have the following natural commutative diagram of functor morphisms:

$$\begin{tikzcd}
f_! \tilde{\phi}_* \tilde{\phi}^* f^! \arrow[d, equal, "f_* = f_!"'] & f_! f^! \arrow[l] \arrow[r] & id_X \arrow[r] & \phi_* \phi^* \arrow[d, equal, "\phi^* = \phi^![-d]"] \\
f_* \tilde{\phi}_* \tilde{\phi}^! f^![-d] = \phi_* \tilde{f}_* \tilde{\phi}^! f^![-d] \underset{\text{base change}}{=} \phi_* \phi^! f_* f^![-d] \arrow[rrr, "f_* = f_!"'] & & & \phi_* \phi^![-d]
\end{tikzcd}$$

Apply it to $\mathbb{D}_X$, we get

$$\begin{tikzcd}
f_! f^! \mathbb{D}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{D}_X \arrow[d] \\
\phi_* \tilde{f}_* \mathbb{D}_{\tilde{Z}}[-d] \arrow[r] & \phi_* \mathbb{D}_{\tilde{X}}[-d]
\end{tikzcd}$$

Take hypercohomology, we get

$$\begin{tikzcd}
H_i(Z) \arrow[r, "f_*"] \arrow[d, "\tilde{\phi}^*"'] \arrow[dr, phantom, "\circlearrowleft"] & H_i(X) \arrow[d, "\phi^*"] \\
H_{i+d}(\tilde{Z}) \arrow[r, "\tilde{f}_*"'] & H_{i+d}(\tilde{X})
\end{tikzcd}$$