1). t-structures.

Recall octahedral axiom: $\mathcal{D}$ is a triangulated cat.

$$\begin{cases} X^\cdot \xrightarrow{f} Y^\cdot \to Z_0^\cdot \to X^\cdot[1] \\ Y^\cdot \xrightarrow{g} Z^\cdot \to X_0^\cdot \to Y^\cdot[1] \\ X^\cdot \xrightarrow{g\circ f} Z^\cdot \to Y_0^\cdot \to X^\cdot[1] \end{cases}$$

three distinguished triangles.

then $\exists$ a distinguished triangle

$$Z_0^\cdot \to Y_0^\cdot \to X_0^\cdot \to Z_0^\cdot[1].$$

Def 1) D a Δ-cat, $D^{\le 0}$, $D^{\ge 0}$ full subcategories,

Set $D^{\le n} := D^{\le 0}[-n]$, $D^{\ge n} := D^{\ge 0}[-n]$, $n \in \mathbb{Z}$. We say the

pair $(D^{\le 0}, D^{\ge 0})$ defines a t-structure on D if

(T1) $D^{\le -1} \subseteq D^{\le 0}$, $D^{\ge 1} \subseteq D^{\ge 0}$

(T2) $\forall X \in D^{\le 0}$, $Y \in D^{\ge 1}$, $\mathrm{Hom}(X, Y) = 0$

(T3) $\forall X \in D$, $\exists$ a distinguished triangle

$$X_0 \to X \to X_1 \xrightarrow{+1}, \quad \text{s.t. } X_0 \in D^{\leq 0} \text{ and } X_1 \in D^{\geq 1}.$$

2) the full subcat. $\mathcal{C} := D^{\geq 0} \cap D^{\leq 0}$ of D

is called the heart of the t-structure.

<u>Example</u>: $\mathcal{A}$ is an abelian cat. $D = D^b(\mathcal{A})$.

$D^{\leq 0} := \{A^\cdot \in D \mid \mathcal{H}^i(A^\cdot) = 0, \ \forall i > 0\}$.

$D^{\geq 0} := \{A^\cdot \in D \mid \mathcal{H}^i(A) = 0, \ \forall i < 0\}$

It's obvious a t-structure, with $D^{\leq 0} \cap D^{\geq 0} = \mathcal{A}$.

<u>Prop</u>: Denote $\iota : D^{\leq n} \to D$ (resp $\iota' : D^{\geq n} \to D$) the inclusion,

Then there exists a functor $\tau^{\leq n} : D \to D^{\leq n}$ (resp.

$\tau^{\geq n} : D \to D^{\geq n}$), s.t. for any $Y \in D^{\leq n}$, $X \in D$ (resp $X \in D$,

$Y \in D^{\geq n}$), $\mathrm{Hom}_{D^{\leq n}}(Y, \tau^{\leq n} X) \xrightarrow{\sim} \mathrm{Hom}_D(\iota(Y), X)$

(resp. $\mathrm{Hom}_{D^{\geq n}}(\tau^{\geq n} X, Y) \xrightarrow{\sim} \mathrm{Hom}_D(X, \iota'(Y))$)

pf: It sufficies to show $\forall X \in D$, $\exists Z \in D^{\leq n}$ and $Z' \in D^{\geq m}$, s.t.

$$\mathrm{Hom}_D(Y, Z) \simeq \mathrm{Hom}_D(Y, X), \quad Y \in D^{\leq n}.$$

$$\mathrm{Hom}_D(Z', Y') \simeq \mathrm{Hom}_D(X, Y'), \quad Y' \in D^{\geq m}.$$

can assume $n=0$, $m=1$. We show $Z = X_0$, $Z' = X_1$ satisfy the property. ($X_0 \to X \to X_1 \xrightarrow{+1}$ in (T3))

$Y \in D^{\leq 0}$, Apply $\mathrm{Hom}_D(Y, \cdot)$ to the dist'd $\triangle$

$$X_0 \to X \to X_1 \xrightarrow{+1}, \quad \text{we get}$$

$$\underset{\substack{\| \,(T2) \\ 0}}{\mathrm{Hom}_D(Y, X_1[-1])} \to \mathrm{Hom}_D(Y, X_0) \to \mathrm{Hom}_D(Y, X) \to \underset{\substack{\| \,(T2) \\ 0}}{\mathrm{Hom}_D(Y, X_1)}$$

$\Rightarrow \mathrm{Hom}_D(Y, X_0) \xrightarrow{\sim} \mathrm{Hom}_D(Y, X)$ $\square$

Rmks:

1) $\tau^{\leq n}$, $\tau^{\geq n}$ truncation functors (unique up to iso as they're adjoint functors)

2) $\exists$ canonical morphisms

$$\tau^{\leq n} X \to X \quad \text{and} \quad X \to \tau^{\geq n} X, \quad X \in D.$$

and $\tau^{\le n}X \to X \to \tau^{\ge n+1}X \xrightarrow{+1}$ is a dist'd $\triangle$.

3) the X_0 and X_1 in (T3) satisfy

$$X_0 \simeq \tau^{\le 0}X, \quad X_1 \simeq \tau^{\ge 1}X.$$

4) $X \in D^{\le n} \iff \tau^{\le n}X \xrightarrow{\sim} X \iff \tau^{>n}X = 0.$

<u>Lemma</u> Let $X' \to X \to X'' \xrightarrow{+1}$ be a dist'd $\triangle$ in D.

If $X', X'' \in D^{\le 0}$ (resp $D^{\ge 0}$), then $X \in D^{\le 0}$ (resp $D^{\ge 0}$).

In particular, if $X', X'' \in \mathcal{C}$, then $X \in \mathcal{C}$.

pf: enough to show $\tau^{>0}X = 0$.

We have exact sequence.

$$\begin{array}{ccccc} \mathrm{Hom}_D(X'', \tau^{>0}X) & \to & \mathrm{Hom}_D(X, \tau^{>0}X) & \to & \mathrm{Hom}_D(X', \tau^{>0}X) \\ \| \ (T2) & & & & \| \ (T2) \\ 0 & & & & 0 \end{array}$$

$$\Rightarrow \quad 0 = \mathrm{Hom}_D(X, \tau^{>0}X) \simeq \mathrm{Hom}_{D^{>0}}(\tau^{>0}X, \tau^{>0}X)$$

$\Rightarrow \tau^{\geq 0} X = 0.$ $\square$

<u>Thm</u>: 1) the heart $\mathcal{C} = D^{\leq 0} \cap D^{\geq 0}$ is an abelian cat.

2) an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$ gives rise to a dist'd $\triangle$ $X \to Y \to Z \xrightarrow{+1}$ in D.

pf: 1) if $X, Y \in \mathcal{C}$, apply the above Lemma to the dist'd $\triangle$

$$X \to X \oplus Y \to Y \xrightarrow{+1},$$ we see $X \oplus Y \in \mathcal{C}$.

Let's show any $f: X \to Y$ admits a kernel and cokernel.

embed f into a dist'd $\triangle$,

$$X \xrightarrow{f} Y \to Z \xrightarrow{+1},$$

then by the above Lemma, $Z \in D^{\leq 0} \cap D^{\geq -1}$.

We'll show that $\operatorname{coker} f \simeq H^0(Z) = \tau^{\geq 0} Z.$

$\ker f \simeq H^{-1}(Z) \simeq \tau^{\leq -1} Z = \tau^{\leq 0}(Z[-1]).$

For any $W \in \mathcal{C}$, consider the exact sequences

$$\begin{array}{ccccccc} \mathrm{Hom}_D(X[1], W) & \to & \mathrm{Hom}_D(Z, W) & \to & \mathrm{Hom}_D(Y, W) & \to & \mathrm{Hom}_D(X, W) \\ \| (T2) & & \| & & & & \\ 0 & & \mathrm{Hom}_{D^{\geq 0}}(\tau^{\geq 0} Z, W) & & & & \\ & & \| & & & & \\ & & \mathrm{Hom}_D(\tau^{\geq 0} Z, W) & & & & \end{array}$$

$\Rightarrow \operatorname{coker} f \simeq \tau^{\geq 0} Z$.

$$\begin{array}{ccccccc} \mathrm{Hom}_D(W, Y[-1]) & \to & \mathrm{Hom}_D(W, Z[-1]) & \to & \mathrm{Hom}_D(W, X) & \to & \mathrm{Hom}_D(W, Y) \\ \| (T2) & & \| & & & & \\ 0 & & \mathrm{Hom}_{D^{\leq 0}}(W, \tau^{\leq 0} Z[-1]). & & & & \end{array}$$

$\Rightarrow \ker f \simeq \tau^{\leq 0} Z[-1]$.

2). Embed $X \xrightarrow{f} Y$ into a dist'd triangle

$$X \xrightarrow{f} Y \to W \xrightarrow{+1}$$

Then. $W \in D^{\leq 0} \cap D^{\geq -1}$.

$\ker f = 0 = \tau^{\geq 0} W \Rightarrow W \in D^{\leq -1} \cap D^{\leq -1}$

$\operatorname{coker} f = Z = \tau^{\leq 0}(W[-1]) = W$. $\square$

Define: $H^0: D \to \mathcal{C} = D^{\leq 0} \cap D^{\geq 0}$

$X \mapsto \tau^{\geq 0}\tau^{\leq 0}X.$

$\forall n \in \mathbb{Z},\ H^n(X) = H^0(X[n])$

Rmk: It's a cohomological functor, i.e. for a dist'd Δ,

$X \to Y \to Z \xrightarrow{+1}$ in D, $H^0(X) \to H^0(Y) \to H^0(Z)$ is exact in $\mathcal{C}$.

Def: D_i Δ-cats with t-structures. $i=1,2$. $F: D_1 \to D_2$.

1) define ${}^pF: \mathcal{C}_1 \to \mathcal{C}_2$ by ${}^pF = H^0 \circ F \circ \varepsilon_1$, $\varepsilon_1: \mathcal{C}_1 \hookrightarrow D_1$.

2) We say F is left t-exact (resp. right t-exact) if

$F(D_1^{\geq 0}) \subseteq D_2^{\geq 0}$ (resp. $F(D_1^{\leq 0}) \subseteq D_2^{\leq 0}$).

We say F is t-exact if it's both left and right t-exact.

Prop: $F: D_1 \to D_2$ is left adjoint to $G: D_2 \to D_1$. Then

F is right t-exact $\Leftrightarrow$ G is left t-exact.

pf: "⇒" Let $Y \in D_2^{\geq 0}$, for any $X \in D_1^{<0}$

$\mathrm{Hom}_{D_1^{<0}}(X, \tau^{<0} G(Y))$

$F(X) \in D_1^{<0}$

$= \mathrm{Hom}_{D_1}(X, G(Y)) = \mathrm{Hom}_{D_2}(F(X), Y) = 0$

$\Rightarrow \tau^{<0} G(Y) = 0 \Rightarrow G(Y) \in D^{\geq 0}. \Rightarrow G$ is left t-exact. $\square$

Prop: $F: D_1 \to D_2$. F left t-exact. Then

1) $\forall X \in D_1$, $\tau^{\leq 0} F \tau^{\leq 0} X \simeq \tau^{\leq 0} F(X)$

2) ${}^pF: \mathcal{C}_1 \to \mathcal{C}_2$ is a left exact functor.

pf: 1) Suffices to show $\forall W \in D_2^{\leq 0}$,

$$\begin{array}{ccc} \mathrm{Hom}_{D_2^{\leq 0}}(W, \tau^{\leq 0} F \tau^{\leq 0} X) & \xrightarrow{\sim} & \mathrm{Hom}_{D_2^{\leq 0}}(W, \tau^{\leq 0} F X) \\ \parallel & & \parallel \\ \mathrm{Hom}_{D_2}(W, F\tau^{\leq 0} X) & & \mathrm{Hom}_{D_2}(W, FX) \end{array}$$

$$\tau^{\leq 0}X \to X \to \tau^{>0}X \xrightarrow{+1} \overset{F}{\Rightarrow} F\tau^{\leq 0}X \to FX \to F\tau^{>0}X \xrightarrow{+1}$$

$$\Rightarrow \underset{\substack{\| \\ 0}}{\mathrm{Hom}_{D_2}(W, F\tau^{>0}X[-1])} \to \mathrm{Hom}_{D_2}(W, F\tau^{\leq 0}X) \xrightarrow{\sim} \mathrm{Hom}_{D_2}(W, FX)$$

$$\to \underset{\substack{\| \\ 0}}{\mathrm{Hom}_{D_2}}(W, \underset{\substack{\cap \\ D_2^{\geq 1}}}{F\tau^{>0}X}) \to \cdots$$

2). $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}_1$

$\rightsquigarrow X \to Y \to Z \xrightarrow{+1}$ in D_1.

$\rightsquigarrow FX \to FY \to FZ \xrightarrow{+1}$ in D_2

$\rightsquigarrow H^{-1}(FZ) \to H^0(FX) \to H^0(FY) \to H^0(FZ)$ in $\mathcal{C}_2$

$FZ \in D_2^{\geq 0} \Rightarrow \tau^{\leq -1}FZ = 0 \Rightarrow H^{-1}(FZ) = \underbrace{(\tau^{\geq -1}\tau^{\leq -1}FZ)[-1]}_{=0}$

$\Rightarrow {}^pF = H^0(F)$ is left exact.

$\square$

2) Perverse sheaves.

X alg variety $/\mathbb{C}$. $D^b_c(X)$ = full subcat of $D^b(Sh(X))$ consisting of constructible complexes.

$D_X = (\ )^\vee$ Verdier duality functor on $D^b_c(X)$.

<u>Def</u> (middle perversity t-structure).

$${}^pD_c^{\leq 0}(X) := \{ F^\cdot \in D^b_c(X) \mid \dim \mathrm{Supp}\, \mathcal{H}^j(F^\cdot) \leq -j,\ \forall j \in \mathbb{Z} \}.$$

$${}^pD_c^{\geq 0}(X) := \{ F^\cdot \in D^b_c(X) \mid \dim \mathrm{Supp}\, \mathcal{H}^j(F^\cdot)^\vee \leq -j,\ \forall j \in \mathbb{Z} \}.$$

<u>Rmk</u>: Since $(F^{\cdot\vee})^\vee \simeq F^\cdot$, the Verdier duality $D_X = (-)^\vee$ exchanges ${}^pD_c^{\leq 0}(X)$ with ${}^pD_c^{\geq 0}(X)$.

<u>Prop</u>: $F^\cdot \in D^b_c(X)$, $X = \bigsqcup X_\alpha$ a stratification with X_α connected, and $i_\alpha^* F^\cdot$ and $i_\alpha^! F^\cdot$ have locally constant cohomology sheaves for any α, where $i_\alpha : X_\alpha \hookrightarrow X$. Then

1) $F^\cdot \in {}^pD_c^{\le 0}(X)$ iff $\mathcal{H}^j(i_\alpha^* F^\cdot) = 0$ $\forall \alpha$ and $j > -d_\alpha := -\dim X_\alpha$

2) $F^\cdot \in {}^pD_c^{\ge 0}(X)$ iff $\mathcal{H}^j(i_\alpha^! F^\cdot) = 0$ $\forall \alpha$ and $j < -d_\alpha$.

pf: 1) is obvious.

2) $\forall x \in X$, $i_x : x \hookrightarrow X$.

$$i_x^! F^\cdot \simeq i_x^! D_X D_X F^\cdot \simeq D_x i_x^* (D_X F^\cdot).$$

$$\Rightarrow \mathcal{H}^{-j}(i_x^! F^\cdot) \simeq \left(\mathcal{H}^j(D_X F^\cdot)_x\right)^*$$

Therefore, $F^\cdot \in {}^pD_c^{\ge 0}(X) \iff \dim\{x \in X \mid \mathcal{H}^{-j}(i_x^! F^\cdot) \neq 0\} \le -j$.

For $x \in X_\alpha$,
$$\begin{tikzcd} x \arrow[r, hook, "j_x"] \arrow[rr, bend right, "i_x"'] & X_\alpha \arrow[r, hook, "i_\alpha"] & X \end{tikzcd}$$

Since $i_\alpha^! F^\cdot$ have locally constant cohomology sheaves,

$$i_x^! F^\cdot \simeq j_x^! i_\alpha^! F^\cdot \simeq j_x^* i_\alpha^! F^\cdot[-2d_\alpha].$$

and X_α is connected, $X_\alpha \cap \{x \in X \mid \mathcal{H}^{-j}(i_x^! F^\cdot) \neq 0\}$ is either X_α or ϕ.

Moreover, the followings are equivalent: (uses the shift $[-2d_\alpha]$).

a) $\mathcal{H}^j(i_\alpha^! F^\cdot) = 0 \quad \forall j < -d_\alpha$

b) $\mathcal{H}^{-j}(i_x^! F^\cdot) = 0 \quad \forall x \in X_\alpha, j > -d_\alpha.$

Hence, $F^\cdot \in {}^pD_c^{\geq 0}(X) \iff \dim\{x \in X \mid \mathcal{H}^{-j}(i_x^! F^\cdot) \neq 0\} \leq -j$

$\iff \mathcal{H}^{-j}(i_x^! F^\cdot) = 0, \forall x \in X_\alpha, j > -d_\alpha.$

$\iff \mathcal{H}^j(i_\alpha^! F^\cdot) = 0 \quad \forall \alpha, j < -d_\alpha.$ $\square$

Cor X connected, $\mathcal{H}^i(F^\cdot)$ are locally constant on X.

then (i) $F^\cdot \in {}^pD_c^{\leq 0}(X)$ iff $\mathcal{H}^j(F^\cdot) = 0 \quad \forall j > -d_X$

(ii) $F^\cdot \in {}^pD_c^{\geq 0}(X)$ iff $\mathcal{H}^j(F^\cdot) = 0 \quad \forall j < -d_X.$

Rmk: $F^\cdot \in D_c^b(X)$,

$F^\cdot \in {}^pD_c^{\geq 0}(X) \iff \forall$ any locally closed subset $S \subseteq X$,

$\mathcal{H}^j(i_S^! F^\cdot) = 0 \quad \forall j < -\dim S.$