Perverse t-structure

<u>Thm</u> $({}^pD_c^{\le 0}(X), {}^pD_c^{\ge 0}(X))$ defines a t-structure on $D_c^b(X)$.

<u>Pf</u>: (T1) $D^{\le -1} \subseteq D^{\le 0}$ is trivial.

For (T2), we prove the following

<u>Lemma</u>: $\forall\ F^\cdot \in {}^pD_c^{\le 0}(X)$ and $G^\cdot \in {}^pD_c^{\ge 0}(X)$,

$$\mathcal{H}^j R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, G^\cdot) = 0 \quad \text{for any } j < 0.$$

(T2) follows from this Lemma.

Pf of the Lemma:

Let $S = \overline{\bigcup_{j<0} \operatorname{Supp} \mathcal{H}^j R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, G^\cdot)} \subseteq X$.

Assume $S \neq \emptyset$. $i_S : S \hookrightarrow X$ closed embedding.

Hence, $\mathcal{H}^j R\mathcal{H}om_{\mathbb{C}_X}(F^\cdot, G^\cdot)$

$\simeq \mathcal{H}^j(i_{S*} i_S^! R\mathcal{H}om(F^\cdot, G^\cdot))$

$\simeq i_{S*}\mathcal{H}^j(R\mathcal{H}om(i_S^*F^\cdot, i_S^! G^\cdot))$

$F^\cdot \in {}^pD_c^{\leq 0}(X) \Rightarrow \dim \operatorname{Supp} \mathcal{H}^k(i_S^* F^\cdot) \leq -k. \quad \forall k \in \mathbb{Z}$

Let $Z := \bigcup_{k > -d_S} \operatorname{Supp} \mathcal{H}^k(i_S^* F^\cdot) \subseteq S.$

then $\dim Z < d_S$

$\Rightarrow S_0 := S \setminus Z \neq \emptyset$, and $\mathcal{H}^j i_{S_0}^* F^\cdot = 0$ for any $j > -d_S$

On the other hand, $\mathcal{H}^j i_S^! G = 0$ for any $j < -d_S$

$\Rightarrow \mathcal{H}^j R\mathcal{H}om_{\mathbb{C}_X}(i_S^* F^\cdot, i_S^! G^\cdot)|_{S_0} = 0$ for any $j < 0$.

But this contradicts the definition of S $\square$

Finally, let's show $\boxed{T3}$.

$F^\cdot \in D_c^b(X)$, take a stratification $X = \bigsqcup X_\alpha$, s.t.

$i_\alpha^* F^\cdot$ and $i_\alpha^! F$ have locally constant cohomology sheaves

for any α. Set $X_k := \bigsqcup_{\dim X_\alpha \leq k} X_\alpha$.

Consider:

$$(S)_k \begin{cases} \exists\, F_0^\cdot \in {}^pD_c^{\le 0}(X\setminus X_k),\ F_1^\cdot \in {}^pD_c^{\ge 1}(X\setminus X_k), \text{ and a dist'd } \triangle \\ F_0^\cdot \to F^\cdot|_{X\setminus X_k} \to F_1^\cdot \xrightarrow{+1} \quad \text{in } D_c^b(X\setminus X_k) \end{cases}$$

We show $(S)_k$ by descending induction on $k \in \mathbb{Z}$.

It's trivial for $k \gg 0$. Assume $(S)_k$ holds, we prove $(S)_{k-1}$

Take $F_0^\cdot \to F^\cdot|_{X\setminus X_k} \to F_1 \xrightarrow{+1}$ as in $(S)_k$.

$$X\setminus X_k \underset{\text{open}}{\overset{\tilde{j}}{\hookrightarrow}} X\setminus X_{k-1} \underset{\text{closed}}{\overset{i}{\hookleftarrow}} X_k\setminus X_{k-1}$$

$F_0 \to F^\cdot|_{X\setminus X_k} \simeq \tilde{j}^!(F|_{X\setminus X_{k-1}}) \simeq \tilde{j}^*(F|_{X\setminus X_{k-1}})$ gives

$$\tilde{j}_! F_0 \to F^\cdot|_{X\setminus X_{k-1}}$$

Embed this into a dist'd $\triangle$

$$\tilde{j}_! F_0 \to F^\cdot|_{X\setminus X_{k-1}} \to G^\cdot \xrightarrow{+1}. \qquad \cdots\cdots \text{①}$$

Also embed $\tau^{\le -k} i_! i^! G^\cdot \to i_! i^! G^\cdot \to G^\cdot$ into a dist'd $\triangle$.

($\tau^{\le -k}$ = usual truncation functor)

$$\tau^{\le -k} i_! i^! G^\cdot \to G^\cdot \to \tilde{F}_1^\cdot \xrightarrow{+1} \qquad \cdots ②$$

Finally, embed $F^\cdot|_{X \setminus X_{k-1}} \to G^\cdot \to \tilde{F}_1^\cdot$ into a dist'd $\triangle$.

$$\tilde{F}_0^\cdot \to F^\cdot|_{X\setminus X_{k-1}} \to \tilde{F}_1^\cdot \xrightarrow{+1} \qquad \cdots ③$$

Hence, we only need to show $\tilde{F}_0 \in {}^pD_c^{\le 0}(X \setminus X_{k-1})$, and

$\tilde{F}_1^\cdot \in {}^pD_c^{\ge 1}(X \setminus X_{k-1})$.

Apply j^* to ②, we get $j^* G^\cdot = j^* \tilde{F}_1^\cdot$

Apply j^* to ①, we get.

$$F_0^\cdot \to F^\cdot|_{X\setminus X_k} \to j^* G^\cdot = j^* \tilde{F}_1 \xrightarrow{+1}$$

Hence, $j^* \tilde{F}_1^\cdot \simeq F_1^\cdot$, and $j^* \tilde{F}_0 = F_0^\cdot$ by ③.

Therefore, we only need to show

(i) $\mathcal{H}^j(i^* \tilde{F}_0^\cdot) = 0 \quad \forall j > -k$

(ii) $\mathcal{H}^j(i^! \tilde{F}_1^\cdot) = 0 \quad \forall j < -k+1$.

Apply the octahedral axiom to

$$\begin{cases} j_! F_0^\cdot \to F^\cdot|_{X\setminus X_{k-1}} \xrightarrow{f} G^\cdot \xrightarrow{+1} \\ \tilde{F}_0^\cdot \to F^\cdot|_{X\setminus X_{k-1}} \xrightarrow{g\circ f} \tilde{F}_1^\cdot \xrightarrow{+1} \\ \tau^{\le -k} i_! i^! G^\cdot \to G^\cdot \xrightarrow{g} \tilde{F}_1^\cdot \xrightarrow{+1} \end{cases}$$

we get

$$j_! F_0^\cdot \to \tilde{F}_0^\cdot \to \tau^{\le -k} i_! i^! G^\cdot \xrightarrow{+1} .$$

Hence, $i^* \tilde{F}_0^\cdot \simeq i^* \tau^{\le -k} i_! i^! G^\cdot \simeq i^* i_! \tau^{\le -k} i^! G^\cdot \simeq \tau^{\le -k} i^! G^\cdot$

The assertion (i) is proved.

Similarly, $i^! \tau^{\le -k} i_! i^! G \simeq \tau^{\le -k} i^! G^\cdot$

$$\left(\begin{array}{c} i^* i_! = \mathrm{Id} \\ \Rightarrow i^! i_* = \mathrm{Id} \\ \shortparallel \\ i^! i_! \end{array} \right)$$

Applying $i^!$ to ②, we get

$$\tau^{\le -k} i^! G \to i^! G^\cdot \to i^! \tilde{F}_1^\cdot \xrightarrow{+1}$$

Hence $i^! \tilde{F}_1^\cdot \simeq \tau^{\ge -k+1} i^! G^\cdot \Rightarrow$ assertion (ii)

$\square$

<u>Def</u> The t-structure $({}^pD_c^{\le 0}(X), {}^pD_c^{\ge 0}(X))$ of $D_c^b(X)$ is called the perverse t-structure (middle pervisity).
An object of its heart $\mathrm{Perv}(\mathbb{C}_X) := {}^pD_c^{\le 0}(X) \cap {}^pD_c^{\ge 0}(X)$ is called a perverse sheaf

$${}^pT^{\le 0} : D^b_c(X) \to {}^pD^{\le 0}_c(X), \quad {}^pT^{\geq 0} : D^b_c(X) \to {}^pD^{\geq 0}_c(X)$$

$\forall n \in \mathbb{Z}$, ${}^pH^n : D^b_c(X) \to Perv(\mathbb{C}_X)$

$$F \mapsto {}^pT^{\le 0}\, {}^pT^{\geq 0}(F^\cdot[n])$$

$\forall F^\cdot \in Perv(X)$, $\mathcal{H}^i(F) = 0$ for $i \notin [-d_X, 0]$

Since $D^b_c(X)^{\le -\dim X} \subseteq {}^pD^{\le 0}_c(X) \Rightarrow {}^pD^{\geq 0}_c(X) \subseteq D^b_c(X)^{\geq -\dim X}$

$\left(Hom(D^b_c(X)^{\le -\dim X - 1}, G^\cdot) = 0 \Rightarrow\right)$

Example: X smooth, $d = \dim X$

1) $\mathbb{C}_X[d] \in Perv(\mathbb{C}_X)$

2) $L \in Loc(X)$, then $L[d] \in Perv(\mathbb{C}_X)$.

$(\mathbb{D}_X(L[d_X]) = R\mathcal{H}om_{\mathbb{C}_X}(L[d], \mathbb{C}_X[2d]) \simeq L^*[d]$.

Prop: i) The Verdier duality $\mathbb{D}_X$ is t-exact, and

induces an exact functor $\mathbb{D}_X : Perv(\mathbb{C}_X) \to Perv(\mathbb{C}_X)$

2) Let $i: Z \hookrightarrow X$ closed embedding. Then $i_*: \mathrm{Perv}(\mathbb{C}_Z) \to \mathrm{Perv}(\mathbb{C}_X)$

pf: 1) is obvious.

2) first of all, $i_* = i_!$ sends ${}^pD_c^{\leq 0}(Z)$ to ${}^pD_c^{\leq 0}(X)$.

$i_* = i_! = D_X \circ i_* \circ D_Z$. Hence, i_* sends ${}^pD_c^{\geq 0}(Z)$ to ${}^pD_c^{\geq 0}(X)$. □

<u>Def</u> $F: D_c^b(X) \to D_c^b(Y)$ induces a functor ${}^pF: \mathrm{Perv}(\mathbb{C}_X) \to \mathrm{Perv}(\mathbb{C}_Y)$

by

$$\begin{tikzcd} \mathrm{Perv}(\mathbb{C}_X) \arrow[r, hook] \arrow[rrr, bend right, "{}^pF"'] & D_c^b(X) \arrow[r, "F"] & D_c^b(Y) \arrow[r, "{}^pH^0"] & \mathrm{Perv}(\mathbb{C}_Y) \end{tikzcd}$$

<u>Prop</u>: $f: Y \to X$, $\dim f^{-1}(x) \leq d \quad \forall x \in X$.

(i) $\forall F^\cdot \in {}^pD_c^{\leq 0}(X)$, $f^*F^\cdot \in {}^pD_c^{\leq d}(Y)$

(ii) $\forall F^\cdot \in {}^pD_c^{\geq 0}(X)$, $f^!F^\cdot \in {}^pD_c^{\geq -d}(Y)$.

Pf: (i) $\dim \mathrm{Supp}\, \mathcal{H}^j(f^*F^\cdot[d]) = \dim f^{-1}(\mathrm{Supp}\, \mathcal{H}^{j+d}(F^\cdot))$

$\leq \dim \mathrm{Supp}\, \mathcal{H}^{j+d}(F^\cdot) + d \leq -j-d+d = -j$.

(ii) follows from (i) as $f^! = D_Y \circ f^* \circ D_X$, and D_X, D_Y are t-exact. $\square$

<u>Cor</u>: 1) $j: U \hookrightarrow X$ open inclusion. Then $j^* = j^!$ is t-exact

2) f smooth of relative dim d, then

$$f^*[d] = f^![-d] \text{ is t-exact.}$$

3) $i: Z \hookrightarrow X$ closed embedding, then $i_* = i_!$ is t-exact.

Moreover, let $\mathrm{Perv}_Z(\mathbb{C}_X) :=$ Perverse sheaves on X whose supp. is contained in Z, then

$$\mathrm{Perv}_Z(\mathbb{C}_X) \underset{{}^p i^{-1} = {}^p i^*}{\overset{i_* = i_!}{\rightleftarrows}} \mathrm{Perv}(\mathbb{C}_Z)$$

is an equivalence.

Examples: 1) $Z \subseteq X$ closed, for any $L \in Loc(Z)$,

$i_{Z*}(L[\dim Z]) \in Perv(\mathbb{C}_X)$

2) X = a connected smooth curve,

$\{x_1, \dots, x_m\}$ points on X. , U = the complement

$X = U \sqcup \{x_1\} \sqcup \{x_2\} \sqcup \dots \sqcup \{x_m\}$ stratification

To visualize an object $\mathcal{F}^\cdot \in D^b_c(X)$, we use the table of stalks

	$i-1$	i	i
U	..	$\mathcal{H}^i(\mathcal{F}\|_U)$	..
x_1	..	$\mathcal{H}^i(\mathcal{F}\|_{x_1})$	..
x_2	..	...	..

we already proved

$${}^pD_c^{\le 0}(X) = \left\{ \mathcal{F} \;\middle|\; \begin{array}{l} \mathcal{H}^i(\mathcal{F}|_U) = 0,\ i > -\dim U = -1 \\ \mathcal{H}^i(\mathcal{F}|_{x_j}) = 0,\ i > 0 \end{array} \right\}$$

$${}^pD_c^{\ge 0}(X) = \left\{ \mathcal{F} \;\middle|\; \begin{array}{l} \mathcal{H}^j(i_U^! \mathcal{F}) = 0,\ j < -1 \\ \mathcal{H}^j(i_{x_k}^! \mathcal{F}) = 0,\ j < 0 \end{array} \right\}, \quad \begin{array}{l} i_U^! = i_U^* \\ i_{x_k}^! \mathcal{F} = i_{x_k}^* \mathcal{F}[-1] \end{array}$$

Hence $\exists$ two possibilities for $\mathcal{F}^{\cdot} \in \mathrm{Perv}(X)$

full support

	-2	-1	0	1
U	0	$\mathcal{L}$	0	0
x_i	0	V	W	0

Skyscraper

	-2	-1	0	1
U	0	0	0	0
x_i	0	0	W'	0

Here are some examples of perverse on X.

- Skyscraper sheaf $\mathbb{C}_{x_j}$
- $\mathcal{L}[1]$ for any $\mathcal{L} \in \mathrm{Loc}(X)$
- $j: U \hookrightarrow X$, $\mathcal{L} \in \mathrm{Loc}(U)$, then $j_!\mathcal{L}[1]$ and $j_*\mathcal{L}[1]$ are perverse.

$j_!$ = extension by 0,

the stalks of $j_! \mathcal{L}[1]$ is

	-2	-1	0	1
U	0	$\mathcal{L}$	0	0
x_i	0	0	0	0

$j_! \mathcal{L}[1] \in {}^pD_c^{\leq 0}(X)$

For $j_* \mathcal{L}[1]$, $x \in X$

$$\mathcal{H}^i(j_* \mathcal{L})_x = \varinjlim_{B(x,\varepsilon)} H^i(B(x,\varepsilon) \cap U, \mathcal{L})$$

$$= \begin{cases} \mathcal{L}_x & \text{if } x \in U, \text{ and } i = 0 \\ H^i(B(x,\varepsilon) \setminus \{x\}, \mathcal{L}) & \text{if } x \notin U \\ 0 & \text{otherwise} \end{cases}$$

$B(x,\varepsilon) \setminus \{x\} \simeq S^1$, $\mathcal{L} \in Loc(U)$, corresponds to a rep V of $\pi_1(U)$.

then $H^i(B(x,\varepsilon) \setminus \{x\}, \mathcal{L})$ can be computed via

$V \xrightarrow{id-u} V$, $u =$ monodromy around x.

$\Rightarrow H^0(S^1, \mathcal{L}) \simeq V^u$, $H^1(S^1, \mathcal{L}) \simeq V_u$ = the coinvariants.

(or first get $H^0(S^1, \mathcal{L}) \simeq V^u$, then duality gets $H^1(S^1, \mathcal{L})$).

Hence, the stalks of $j_*\mathcal{L}[1]$ is

	-2	-1	0	1
U	0	$\mathcal{L}$	0	0
x_i	0	V^u	V_u	0

,

$$j_*\mathcal{L}[1] \in {}^pD_c^{\leq 0}(X).$$

Since $D_X(j_*\mathcal{L}[1]) = j_! D_U \mathcal{L}[1]$

$j_*\mathcal{L}[1]$ and $j_!\mathcal{L}[1]$ are both perverse.

i.e. j_* and $j_!$ are t-exact.

We will come back to this example later when we study intersection cohomology.